Homework for March 29, 2020.

Algebra.

Read the classwork handout. Complete the unsolved problems from the previous homework. Solve the following problems. As usual, you do not necessarily have to solve every problem. However, please solve as much as you can in the time you have. Start with those that "catch your eye", in one way or another (e.g. you think are the easiest, or most challenging). Skip the ones you already solved in the past.

1. Perform long division of the following polynomials.
   a. $(x^5 - 2x^3 + 3x^2 - 4) \div (x^2 - x + 1)$
   b. $(x^4 - x^2 + 1) \div (x + 1)$
   c. $(x^7 + 1) \div (x^3 - x + 1)$
   d. $(6x^6 - 5x^5 + 4x^4 - 3x^3 + 2x - 1) \div (x^2 + 1)$
   e. $(x^5 - 32) \div (x + 2)$
   f. $(x^5 - 32) \div (x - 2)$
   g. $(x^6 + 64) \div (x^2 + 4)$
   h. $(x^6 + 64) \div (x^2 - 4)$
   i. $(x^{100} - 1) \div (x^2 - 1)$
2. Can you find coefficients a, b, such that there is no remainder upon division of a polynomial, $x^4 + ax^3 + bx^2 - 2x - 10$,
   a. by $x + 5$
   b. by $x^2 + x - 1$
3. Prove that,
   a. for odd n, the polynomial $x^n + 1$ is divisible by $x + 1$
   b. $2^{100} + 1$ is divisible by 17.
   c. $2^n + 1$ can only be prime if n is a power of 2 [Primes of this form are called Fermat primes; there are very few of them. How many can you find?]
   d. for any natural number n, $8^n - 1$ is divisible by 7.
   e. for any natural number n, $15^n + 6$ is divisible by 7
4. Factor (i.e., write as a product of polynomials of smaller degree) the following polynomials.

   a. $1 + a + a^2 + a^3$
   b. $1 - a + a^2 - a^3 + a^4 - a^5$
   c. $a^3 + 3a^2b + 3b^2a + b^3$

d. $x^4 - 3x^2 + 2$

5. Simplify the following expressions using polynomial factorization.

   e. $\frac{x+y}{x} - \frac{x}{x-y} + \frac{y^2}{x^2-xy}$

   f. $\frac{x^6-1}{x^4+x^2+1}$

   g. $\frac{a^3-2a^2+5a+26}{a^3-5a^2+17a-13}$

6. Solve the following equations

   h. $\frac{x^2+1}{x} + \frac{x}{x^2+1} = 2.9$ (hint: substitution)

   i. $\frac{14}{20-6x-2x^2} + \frac{x^2+4x}{x^2+5x} - \frac{x+3}{x-2} + 3 = 0$ (hint: factorize square polynomials)

7. Write Vieta formulae for the reduced cubic equation, $x^3 + px + q = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of p and q,

   j. $(x_1 + x_2 + x_3)^2$

   k. ${x_1}^2 + {x_2}^2 + {x_3}^2$

   l. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$

   m. $(x_1 + x_2 + x_3)^3$

8. The three real numbers x, y, z, satisfy the equations

$$x + y + z = 7$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{7}$$

   Prove that then, at least one of x, y, z is equal to 7. [Hint: Vieta formulas]

9. Find all real roots of the following polynomial and factor it: $x^4 - x^3 + 5x^2 - x - 6$.

Algebra/Trigonometry.

Please, complete the previous homework assignments from this year. Review the classwork handout on trigonometric functions. Additional reading on trigonometric functions is Read Gelfand & Saul, Trigonometry, Chapter 2 (pp. 42-54) and Chapters 6-8 (pp. 123-163),
http://en.wikipedia.org/wiki/Trigonometric_functions
http://en.wikipedia.org/wiki/Sine. Solve the following problems.

Problems.

1. Find the sum of the following series,

$$S = \cos x + \cos 2x + \cos 3x + \cos 4x + \cdots + \cos Nx$$

(hint: multiply the sum by $2 \sin x/2$)

2. Find all x for which,
   a. $\sin x \cos x = \frac{1}{2}$
   b. $\sin x \cos x = \frac{\sqrt{3}}{2}$
3. Simplify the following expression:
   a. $(1 + \sin \alpha)(1 - \sin \alpha)$
   b. $(1 + \cos \alpha)(1 - \cos \alpha)$
   c. $\sin^4 \alpha - \cos^4 \alpha$
   d. $\cos^2 \left(\alpha - \frac{\pi}{6}\right) + \cos^2 \left(\alpha + \frac{\pi}{6}\right) + \sin^2 \alpha =$
4. Calculate:
   a. $\cos 75° + \cos 15° =$
   b. $\cos \frac{\pi}{12} - \cos \frac{5\pi}{12} =$
5. Solve the following equation:
   a. $\cos^2 \pi x + 4 \sin \pi x + 4 = 0$
6. Prove the following equalities:

a. $\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha} = \cot\frac{\alpha}{2}$

b. $\sin^2\left(\frac{7\pi}{8} - 2\alpha\right) - \sin^2\left(\frac{9\pi}{8} - 2\alpha\right) = \frac{\sin 4\alpha}{\sqrt{2}}$

c. $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}$

d. $\frac{\cot^2 2\alpha - 1}{2\cot 2\alpha} - \cos 8\alpha \cot 4\alpha = \sin 8\alpha$

e. $\sin^6\alpha + \cos^6\alpha + 3\sin^2\alpha\cos^2\alpha = 1$

f. $\frac{\sin 6\alpha + \sin 7\alpha + \sin 8\alpha + \sin 9\alpha}{\cos 6\alpha + \cos 7\alpha + \cos 8\alpha + \cos 9\alpha} = \tan\frac{15\alpha}{2}$

g. $\sin^6\alpha + \cos^6\alpha = \frac{5 + 3\cos 4\alpha}{8}$

h. $16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha = \sin 5\alpha$

i. $\frac{\cos 64^\circ \cos 4^\circ - \cos 86^\circ \cos 26^\circ}{\cos 71^\circ \cos 41^\circ - \cos 49^\circ \cos 19^\circ}$

j. $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$

k. $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$

7. Simplify the following expressions:

a. $\sin^2\left(\frac{\alpha}{2} + 2\beta\right) - \sin^2\left(\frac{\alpha}{2} - 2\beta\right)$

b. $2\cos^2 3\alpha + \sqrt{3}\sin 6\alpha - 1$

c. $\cos^4 2\alpha - 6\cos^2 2\alpha \sin^2 2\alpha + \sin^4 2\alpha$

d. $\sin^2(135^\circ - 2\alpha) - \sin^2(210^\circ - 2\alpha) - \sin 195^\circ \cos(165^\circ - 4\alpha)$

e. $\frac{\cos 2\alpha - \cos 6\alpha + \cos 10\alpha - \cos 14\alpha}{\sin 2\alpha + \sin 6\alpha + \sin 10\alpha + \sin 14\alpha}$

8. Find the period of the function $y = \sin 5x - 2\sin 7x$

9. Let A, B and C be angles of a triangle. Prove that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

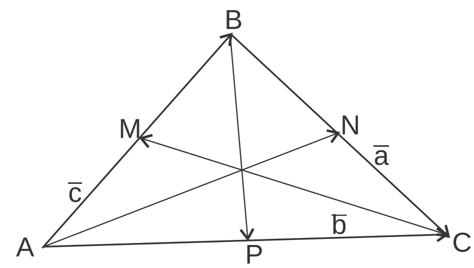
10. Solve the following equations and inequalities:

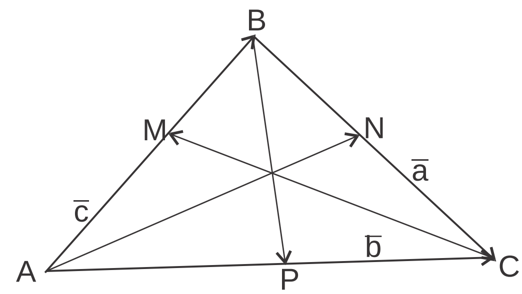
f. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$
g. $\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)$
h. $\sin^2 x - 2 \sin x \cos x = 3 \cos^2 x$
i. $\sin 6x + 2 = 2 \cos 4x$
j. $\cot x - \tan x = \sin x + \cos x$
k. $\sin x \geq \pi/2$
l. $\sin x \leq \cos x$

Geometry. Vectors.

Please, complete the previous homework assignments from this year. Review the classwork handout on vectors. Solve the following problems.

Problems.





1. In a triangle ABC, vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$and $\overrightarrow{BC}$ (**c**, **b** and **a**) are the sides. $\overrightarrow{AN}$, $\overrightarrow{CM}$and $\overrightarrow{BP}$are the medians.

   a. Express vectors $\overrightarrow{AN}$, $\overrightarrow{CM}$and $\overrightarrow{BP}$ through vectors **c**, **b** and **a.**

   b. Find the sum of vectors $\overrightarrow{AN}$, $\overrightarrow{CM}$and $\overrightarrow{BP}$**.**

2. Solve the same problem for bisectors $\overrightarrow{AN}$, $\overrightarrow{CM}$and $\overrightarrow{BP}$ in a triangle ABC.

3. Coxeter, Greitzer, problem #9 to Sec. 2.1 (p. 31): How far away is the horizon as seen from the top of a mountain 1 mile high? (Assume the Earth to be a sphere of diameter 7920 miles.)

4. In a rectangle $ABCD$, A_1, B_1, C_1 and D_1 are the mid-points of sides AB, CD, BC and DA, respectively. M is the crossing point of the segments A_1B_1, and C_1D_1, connecting two pairs of midpoints.

   a. Express vector $\overrightarrow{A_1M}$ through $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$.

   b. Prove that M is the mid-point of segments, A_1B_1 and C_1D_1, i.e. $|A_1M| = |MB_1|$ and $|C_1M| = |MD_1|$.

5. In a parallelogram $ABCD$, find $\overrightarrow{AB} + \overrightarrow{BD} - 2\overrightarrow{AD}$.

6. M is a crossing point of the medians in a triangle ABC. Prove that $\overrightarrow{AM} = \frac{1}{3}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$.

7. For three points, $A(-1,3)$, $B(2,-5)$ and $C(3,4)$, find the (coordinates of) following vectors,

   a. $\overrightarrow{AB} - \overrightarrow{BC}$

   b. $\overrightarrow{AB} + \overrightarrow{CB} + \overrightarrow{AC}$

c. $\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} + \frac{1}{3}\overrightarrow{CA}$

8. For two triangles, ABC and $A_1B_1C_1$, $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = 0$. Prove that medians of these two triangles cross at the same point M.