Homework for March 15, 2020.

Algebra.

Review the classwork handout and complete the exercises which were not solved in class. Try solving the unsolved problems from the previous homework (some are repeated below) and the following new problems.

1. Prove the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

Ordering and comparison.

a. $\forall\, a, b \in \mathbb{R}$, one and only one of the following relations holds
   i. $a = b$
   ii. $a < b$
   iii. $a > b$
b. $\forall\, a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
c. Transitivity. $\forall\, a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
d. Archimedean property: $\forall\, a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}: a < nb$

Addition and subtraction.

a. $\forall\, a, b \in \mathbb{R}, a + b = b + a$
b. $\forall\, a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
c. $\forall\, a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
d. $\forall\, a \in \mathbb{R}, \exists -a \in \mathbb{R}, a + (-a) = 0$
e. $\forall\, a, b \in \mathbb{R}, a - b = a + (-b)$
f. $\forall\, a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$

2. Show that for the set of real numbers, $\mathbb{R}$, cardinality of the set of all possible subsets is greater than that of a continuum of real numbers itself.
3. Prove that the cardinality of the set of all points on a sphere is the same as that of the set of all points on a circle.
4. Represent $\sqrt{2}$ (and $\sqrt{p}$ for any rational p) by using the continuous fraction,

$$\sqrt{2} = a + \cfrac{c}{b + \cfrac{c}{b + \cfrac{c}{b + \cdots}}}$$

Geometry/Trigonometry.

Read the classwork handout. Complete the unsolved problems from the previous homework and classwork exercises. Additional reading on trigonometric functions is Gelfand & Saul, Trigonometry, Chapter 2 (pp. 42-54) and Chapters 6-8 (pp. 123-163),
http://en.wikipedia.org/wiki/Trigonometric_functions
http://en.wikipedia.org/wiki/Sine. Solve the following problems.

1. Using the expressions for the sine and the cosine of the sum of two angles derived in class, derive expressions for:
   a. $\sin 3\alpha$
   b. $\cos 3\alpha$
   c. $\tan(\alpha \pm \beta)$
   d. $\cot(\alpha \pm \beta)$
   e. $\tan(2\alpha)$
   f. $\cot(2\alpha)$
2. Derive the following expressions (classwork exercise):
   a. $\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos(\alpha)\cos(\beta)}$ $\quad\cot\alpha \pm \cot\beta = \pm\frac{\sin(\alpha\pm\beta)}{\sin(\alpha)\sin(\beta)}$
   b. $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$ $\quad\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$
   c. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ $\quad\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$
   d. $\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$ $\quad\cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$
   e. $\sin\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1-\cos\alpha)}$ $\quad\cos\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1+\cos\alpha)}$
   f. $\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$
   g. $\cot\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{1-\sin\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}$
   h. $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$
   i. $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$
   j. $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

3. Show that the length of a chord in a circle of unit diameter is equal to the sine of its inscribed angle.

4. Using the result of the previous problem, express the statement of the Ptolemy theorem in the trigonometric form, also known as Ptolemy identity (see Figure):

$$\sin(\alpha+\beta)\sin(\beta+\gamma) = \sin\alpha\sin\gamma + \sin\beta\sin\delta,$$

if $\alpha+\beta+\gamma+\delta = \pi$.

5. Prove the Ptolemy identity in Problem 2 using the addition formulas for sine and cosine.

6. Using the Sine and the Cosine theorems, prove the Heron's formula for the area of a triangle,

$$S_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$ is the semi-perimeter.

7. Show that

a. $\cos^2\alpha + \cos^2\left(\frac{2\pi}{3}+\alpha\right) + \cos^2\left(\frac{2\pi}{3}-\alpha\right) = \frac{3}{2}$

b. $\sin\alpha + \sin\left(\frac{2\pi}{3}+\alpha\right) + \sin\left(\frac{4\pi}{3}+\alpha\right) = 0$

c. $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$

8. Without using calculator, find:

a. $\sin 75° =$

b. $\cos 75° =$

c. $\sin\frac{\pi}{8} =$

d. $\cos\frac{\pi}{8} =$

e. $\sin\frac{\pi}{16} =$

f. $\cos\frac{\pi}{16} =$