Homework for March 8, 2020.

Algebra.

Review the classwork handout. Try solving the unsolved problems from the previous homework. Review the classwork exercises and complete those which were not solved (some are repeated below). Solve the following problems.

1. Assume that the set of rational numbers $\mathbb{Q}$ is divided into two subsets, $\mathbb{Q}_<$ and $\mathbb{Q}_>$, such that all elements of $\mathbb{Q}_>$ are larger than any element of $\mathbb{Q}_<$: $\forall a \in \mathbb{Q}_<, \forall b \in \mathbb{Q}_>, a < b$.
   a. Prove that if $\mathbb{Q}_>$ contains the smallest element, $\exists b_0 \in \mathbb{Q}_>, \forall b \in \mathbb{Q}_>, b_0 \leq b$, then $\mathbb{Q}_<$ does not contain the largest element
   b. Prove that if $\mathbb{Q}_<$ contains the largest element, $\exists a_0 \in \mathbb{Q}_<, \forall a \in \mathbb{Q}_<, a \leq a_0$, then $\mathbb{Q}_>$ does not contain the smallest element
   c. Present an example of such a partition, where neither $\mathbb{Q}_>$ contains the smallest element, nor $\mathbb{Q}_<$ contains the largest element
2. Prove the following properties of countable sets. For any two countable sets, A, B,
   a. Union, $A \cup B$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big)$ $\Rightarrow (c(A \cup B) = \aleph_0)$
   b. Product, $A \times B = \{(a, b), a \in A, b \in B\}$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
   c. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 \ldots \cup A_n) = \aleph_0$
3. Prove that the following numbers are irrational:
   a. $\sqrt[2]{2}$
   b. $\sqrt[3]{3}$
   c. $\sqrt[5]{5}$
4. Compare the following real numbers (are they equal? which is larger?)
   a. $1.33333\ldots = 1.(3)$ and $4/3$
   b. $0.09999\ldots = 0.0(9)$ and $1/10$
   c. $99.9999\ldots = 99.(9)$ and 100
   d. $\sqrt[2]{2}$ and $\sqrt[3]{3}$

5. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
    a. $1/8$
    b. $2/7$
    c. 0.1
    d. $0.33333\ldots = 0.(3)$
    e. $0.13333\ldots = 0.1(3)$
6. * Let W be the set of all "words" that can be written using the alphabet consisitng of 26 lowercase English letters; by a "word", we mean any (finite) sequence of letters, even if it makes no sense – for example, abababaaaaa. Prove that W is countable. [Hint: for any n, there are only finitely many words of length n.]
7. Try proving the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

Ordering and comparison.

1. $\forall\ a, b \in \mathbb{R}$, one and only one of the following relations holds
    - $a = b$
    - $a < b$
    - $a > b$
2. $\forall\ a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\ a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\ a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}$, such that $a < nb$

Addition and subtraction.

- $\forall\ a, b \in \mathbb{R}, a + b = b + a$
- $\forall\ a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
- $\forall\ a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
- $\forall\ a \in \mathbb{R}, \exists -a \in \mathbb{R}, a + (-a) = 0$
- $\forall\ a, b \in \mathbb{R}, a - b = a + (-b)$
- $\forall\ a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$

Geometry/Trigonometry.

Read the classwork handout. Complete the unsolved problems from the previous homework and classwork exercises. Additional reading on trigonometric functions is Gelfand & Saul, Trigonometry, Chapter 2 (pp. 42-54) and Chapters 6-8 (pp. 123-163),
http://en.wikipedia.org/wiki/Trigonometric_functions
http://en.wikipedia.org/wiki/Sine. Solve the following problems.

1. Find the distance between the nearest points of the circles,
   a. $(x-2)^2 + y^2 = 4$ and $x^2 + (y-1)^2 = 9$
   b. $(x+3)^2 + y^2 = 4$ and $x^2 + (y-4)^2 = 9$
   c. $(x-2)^2 + (y+1)^2 = 4$ and $(x+1)^2 + (y-3)^2 = 5$
   d. $(x-a)^2 + y^2 = r_1^2$ and $x^2 + (y-b)^2 = r_2^2$
2. Using the expressions for the sine and the cosine of the sum of two angles derived in class, derive expressions for (classwork exercise),
   a. $\sin 3\alpha$
   b. $\cos 3\alpha$
   c. $\tan(\alpha \pm \beta)$
   d. $\cot(\alpha \pm \beta)$
   e. $\tan(2\alpha)$
   f. $\cot(2\alpha)$