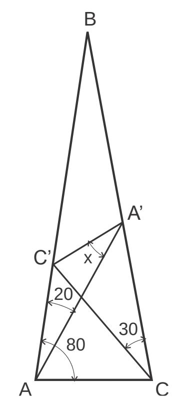
Homework for January 17, 2020.

Geometry.

Review the previous classwork notes. Solve the remaining problems from the previous homework (some are below). Solve the following problems.

Problems.



1. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 30°$ and $\angle BAA' = 20°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.
2. (Skanavi 15.104) Given the right triangle ABC ($\angle C = 90°$), find the locus of points $P(x, y)$ such that $|PA|^2 + |PB|^2 = 2|PC|^2$.
3. (Skanavi 15.109) Points $A(-1,2)$ and $B(4, -2)$ are vertices of the rhombus $ABCD$, while point $M(-2,0)$ belongs to the side CD. Find the coordinates of the vertices C and D.
4. (Skanavi 15.114) Find the circle (write the equation of this circle) passing through the coordinate origin, $O(0,0)$, point $A(1,0)$ and tangent to the circle $x^2 + y^2 = 9$.
5. (Skanavi 15.115) Write the equation of the circle passing through the point $A(2,1)$ and tangent to both X- and Y-axes.
6. *Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.
7. *Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle.

8. Using the Ptolemy's theorem, prove the following:
   a. Given an equilateral triangle ΔABC inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of

the triangle is the sum of the distances from the point to the two nearer vertices.

b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.

Algebra.

Review the classwork handout and complete the exercises that were not solved (some are repeated below). Solve the following problems, including problems from previous home works, some of which are repeated below.

1. Prove the following properties of countable sets. For any two countable sets, A, B,
   a. Union, $A \cup B$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
   b. Product, $A \times B = \{(a, b), a \in A, b \in B\}$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
   c. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 \ldots \cup A_n) = \aleph_0$
2. Let W be the set of all "words" that can be written using the alphabet consisitng of 26 lowercase English letters; by a "word", we mean any (finite) sequence of letters, even if it makes no sense – for example, abababaaaaa. Prove that W is countable. [Hint: for any n, there are only finitely many words of length n.]
3. Show that the only possible remainders of division of the square of a natural number, n^2, by 3 are 0 and 1. What are the possible remainders of division of the square of a natural number, n^2, by 7?
4. * If 9 dies are rolled, what is the probability that all 6 numbers appear?
5. * How many permutations of the 26 letters of English alphabet do not contain any of the words *pin*, *fork*, or *rope*?