Homework for December 13, 2019.

Algebra.

Review the classwork handout. Review and solve the classwork exercises which were not solved (some are repeated below). Solve the following problems.

1. Present examples of binary relations that are, and that are not equivalence relations.
2. For each of the following relations, check whether it is an equivalence relation and describe all equivalence classes.
   a. On $\mathbb{R}$: relation given by $x \sim y$ if $|x|=|y|$
   b. On $\mathbb{Z}$: relation given by $a \sim b$ if $a \equiv b \ mod \ 5$
   c. On $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 + x_2 = y_1 + y_2$; describe the equivalence class of $(1, 2)$
   d. Let $\sim$ be the relation on the set of all directed segments in the plane defined by $\overrightarrow{A\,B} \sim \overrightarrow{A'\,B'}$ if $ABB'A'$ is a parallelogram.
   e. On the set of pairs of integers, $\{(a, b), a, b \in \mathbb{Z}, b \neq 0\}$, $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 b_2 = a_2 b_1$. Describe these equivalence classes. Is the set of the obtained equivalence classes countable?
3. Let $f: X \xrightarrow{f} Y$ be a function. Define a relation on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Prove that it is an equivalence relation. Describe the equivalence classes for the equivalences defined by the following functions on $\mathbb{R}$.
   a. $f(x) = x^2$: $x \sim y$ if $x^2 = y^2$.
   b. $f(x) = \sin x$: $x \sim y$ if $\sin x = \sin y$.
4. Find the following sum.

$$\left(2 + \frac{1}{2}\right)^2 + \left(4 + \frac{1}{4}\right)^2 + \cdots + \left(2^n + \frac{1}{2^n}\right)^2$$

5. The lengths of the sides of a triangle are three consecutive terms of the geometric series. Is the common ratio of this series, q, larger or smaller than 2?
6. Solve the following equation,

$\frac{x-1}{x} + \frac{x-2}{x} + \frac{x-3}{x} + \cdots + \frac{1}{x} = 3$, where x is a positive integer.

7. Find the following sum,
    a. $1 + 2 \cdot 3 + 3 \cdot 7 + \cdots + n \cdot (2^n - 1)$
    b. $1 \cdot 3 + 3 \cdot 9 + 5 \cdot 27 + \cdots + (2n - 1) \cdot 3^n$
8. Numbers $a_1, a_2, \dots, a_n$ are the consecutive terms of a geometric progression, and the sum of its first n terms is S_n. Show that,

$$S_n = a_1 a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

9. Prove that three terms shown below are the three terms of the geometric progression, and find the sum of its first n terms, beginning with the first one below,

$$\frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{1}{3-\sqrt{3}} + \frac{1}{6} + \cdots$$

10. What is the maximum value of the expression, $(1 + x)^{36} + (1 - x)^{36}$ in the interval $|x| \leq 1$?
11. Find the coefficient multiplying x^9 after all parenthesis are expanded in the expression, $(1 + x)^9 + (1 + x)^{10} + \cdots + (1 + x)^{19}$.