November 17, 2019

Algebra.

Cartesian product.

Given two sets, A and B, we can construct a third set, C, which is made of all possible ordered pairs of the elements of these sets, (a, b), where $a \in A$ and $b \in B$. We thus have a **binary operation**, which acts on a pair of objects (sets A and B) and returns a third object (set C). Following Rene Descartes, who first considered such construction in the context of Cartesian coordinates of points on a plane, in mathematics such operation is called Cartesian product.

A **Cartesian product** is a mathematical operation that returns a (product) set from multiple sets. For two sets A and B, the Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$,

$$A \times B = \{(a, b): a \in A \ \wedge b \in B\}$$

Example 1. A table can be created from a single row and a single column, by taking the Cartesian product of a set of objects in a row and a set of objects in a column. In the Cartesian product row × column, the cells of the table contain ordered pairs of the form (row object, column object).

Example 2. Another example is a 52 (or 36) card deck. In a 52 card deck, the standard playing card ranks {A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2} form a 13-element set. The card suits {♠, ♥, ♦, ♣} form a four-element set. The Cartesian product of these two sets returns a 52-element set consisting of 52 ordered pairs, which correspond to all 52 possible playing cards. Ranks × Suits returns a set of the form {(A, ♠), (A, ♥), (A, ♦), (A, ♣), (K, ♠), ..., (3, ♣), (2, ♠), (2, ♥), (2, ♦), (2, ♣)}. Suits × Ranks returns a set of the form {(♠, A), (♠, K), (♠, Q), (♠, J), (♠, 10), ..., (♣, 6), (♣, 5), (♣, 4), (♣, 3), (♣, 2)}. Are these two sets different?

The Cartesian product A × B is **not commutative**, because the elements in the ordered pairs are reversed.

$$\{(a, b): a \in A \ \wedge b \in B\} = A \ \times \ B \ \neq \ B \ \times \ A = \{(b, a): a \in A \ \wedge b \in B\}$$

Exercise 1. Construct Cartesian product for sets:

- $A = \{13,14\}; B = \{1,1\}$
- $A = \{3,5,7\}; B = \{7,5,3\}$
- $A = \{a,b,c,d,e,f,g,h\}; B = \{1,2,3,4,5,6,7,8\}$
- $A = \{J,F,M,A,M,J,J,A,S,O,N,D\}; B = \{n: n \in \mathbb{N} \wedge n \leq 31\}$

The Cartesian product is **essentially associative**: while it is not literally true that $(A \times B) \times C = A \times (B \times C)$, there is an obvious way to identify these two sets by identifying $\big((a,b),c\big)$ with $\big(a(b,c)\big)$ (for those who want absolute mathematical rigor, we can say that there is an obvious bijective correspondence between these two sets – we will discuss bijections in more detail later).

In a similar way, one can define cartesian product of n sets

$$A_1 \times A_2 \times \ldots \times A_n$$

as the set of (ordered) n-tuples $(a_1, \ldots a_n), a_i \in A_i$.

The Cartesian product has the following property with respect to intersections,

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$$

The above statement is not true if we replace intersection with union,

$$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$$

Exercise 2. Prove the following distributivity properties of Cartesian products,

- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

Exercise 3. Present several examples of Cartesian products of sets, which are encountered in everyday life. Describe (define) the constituent sets. Discuss how properties listed above are manifest in the resultant Cartesian products.

Solutions to some homework problems.

1. **Problem**. Using the method of mathematical induction, prove the following equalities,

$$\sum_{k=0}^{n} k\cdot k! = (n+1)! - 1$$

Solution. Base: $\sum_{k=0}^{1} k\cdot k! = 0 + 1 = (1+1)! - 1$

Induction: $\sum_{k=0}^{n+1} k\cdot k! = \sum_{k=0}^{n+1} k\cdot k! + (n+1)\cdot(n+1)! = (n+1)! - 1 + (n+1)\cdot(n+1)! = (n+1)!\cdot(n+2) - 1 = (n+2)! - 1$

2. **Problem**. Put the sign $<, >$, or $=$, in place of … below,

$$\frac{n+1}{2} \ldots \sqrt[n]{n!}$$

Solution. Consider set of n consecutive integers, $\{1,2,3,\ldots n\}$. Using the inequality between the arithmetic and the geometric mean of these n numbers we obtain,

$$\frac{1+2+\cdots+n}{n} \geq \sqrt[n]{1\cdot 2\cdot \ldots \cdot n} \Leftrightarrow \frac{n(n+1)}{2n} = \frac{n+1}{2} \geq \sqrt[n]{n!}$$

3. **Problem**. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.

Solution. From $x^2 = ax + b \Leftarrow x = a + \frac{b}{x}$, we find $x = \{7,7,7,\ldots\} = 7 + \frac{1}{7+\frac{1}{7+\frac{1}{7+\cdots}}}$.

4. **Problem**. Find the value of the continued fraction given by $x = \{1,2,3,3,3,\ldots\}$.

Solution. $x = 1 + \frac{1}{y-1}$, $y = \{3,3,3,\ldots\} = 3 + \frac{1}{3+\frac{1}{3+\frac{1}{3+\cdots}}} \Rightarrow y^2 - 3y - 1 = 0$

$\Leftrightarrow y = \frac{3+\sqrt{13}}{2}$.