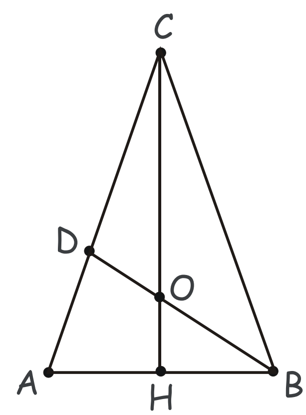
Homework for November 3, 2019.

Geometry.

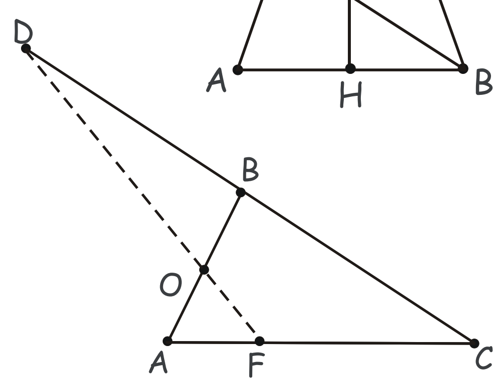
Review the classwork handout. Solve the unsolved problems from previous homeworks. Try solving the following problems. Use the similarity of triangles and Thales theorem. In the next homework, you will solve some of these problems using the method of point masses and the Law of Lever.

Problems.

1. Prove that for any triangle ABC with sides a, b and c, the area, $S \leq \frac{1}{4}(b^2 + c^2)$.
2. Prove that if a polygon has several axes of symmetry, they are all concurrent (cross at the same point).
3. Prove that medians of a triangle divide one another in the ratio 2:1, in other words, the medians of a triangle "trisect" one another (Coxeter, Gretzer, p.8).
4. In isosceles triangle ABC point D divides the side AC into segments such that |AD|:|CD|=1:2. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio |OH| to |CH|.



5. Point D belongs to the continuation of side CB of the triangle ABC such that |BD| = |BC|. Point F belongs to side AC, and |FC| = 3|AF|. Segment DF intercepts side AB at point O. Find the ratio |AO|:|OB|.



Algebra.

Review the classwork handout and complete the exercises. Solve the remaining problems from the previous homework (you may skip the ones considered in class). Solve the following problems.

1. Using mathematical induction, prove that $\forall n \in \mathbb{N}$,
   a. $\sum_{k=1}^{n}(2k-1)^2 = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3-n}{3}$,
   b. $\sum_{k=1}^{n}(2k)^2 = 2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n(2n+1)(n+1)}{3}$
   c. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2$
   d. $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} < \frac{1}{2}$
   e. $\sum_{k=1}^{n} \frac{1}{(7k-6)(7k+1)} = \frac{1}{1\cdot 8} + \frac{1}{8\cdot 15} + \frac{1}{15\cdot 22} + \cdots + \frac{1}{(7n-6)(7n+1)} < \frac{1}{7}$
   f. $\sum_{k=n+1}^{3n+1} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n+1} > 1$
2. **Recap.** Binomial coefficients are defined by
   $C_n^k = {}_kC_n = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   a. Prove that $C_{n+k}^2 + C_{n+k+1}^2$ is a full square
   b. Find n satisfying the following equation,

   $$C_n^{n-1} + C_n^{n-2} + C_n^{n-3} + \cdots + C_n^{n-10} = 1023$$

   c. Prove that

   $$\frac{C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n}{n} = 2^{n-1}$$
3. Find all x satisfying the following equation:

$$\frac{14}{20-6x-2x^2} + \frac{x^2+4x}{x^2+5x} - \frac{x+3}{x-2} + 3 = 0.$$