MATH 8: ASSIGNMENT 18

FEBRUARY 9TH, 2020

1. Points and Lines

Axiom 1 (Lines)**.** *For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

Axiom 2 (Distance)**.** *If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$.*

Axiom 3 (Angles)**.** *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*

Axiom 4 (Euclid's Parallel Postulate)**.** *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called* alternate interior angles*). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*

Theorem 1. *If lines l, m intersect, than they intersect at exactly one point.*

Theorem 2 (Parallelism)**.** *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$.*

Theorem 3 (Vertical Angles)**.** *Let M be the intersection point of line segments $\overline{AB}$ and $\overline{CD}$. Then $m\angle AMC = m\angle BMD$ (such a pair of angles are called* vertical*).*

Theorem 4 (Perpendicularity)**.** *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are* perpendicular *and write $l \perp m$.)*

Theorem 5. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$. Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.*

Theorem 6. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

2. Triangles and Congruence

Definition 1 (Triangles)**.** A triangle consists of three points (called the vertices of the triangle) and their three corresponding line segments (the sides of the triangle). A triangle with vertices A, B, C is written $\triangle ABC$, and has sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, and interior angles $\angle ABC$, $\angle BCA$, $\angle CAB$.

Definition 2 (Altitude, Median, Angle Bisector)**.** In triangle $\triangle ABC$,

- The altitude from A is perpendicular to BC (it exists and is unique by Theorem 7).
- The median from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC).
- The angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

Definition 3 (Isosceles Triangles)**.** A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

Definition 4 (Congruence)**.** Congruence between two objects of the same type indicates the following:

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D, then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong CD$.
- If two triangles $\triangle ABC$, $\triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Definition 5 (Locus)**.** A locus is the set of all points satisfying a certain property.

The concept of a locus is quite versatile; we will use it to describe lines and circles, but other figures can be described as well. For example, given two points A, B, the locus of points P whose sum of distances to A, B is a fixed constant (i.e. $AP + BP = c$ for some c) is called an *ellipse*. But that is a fun fact, and we won't dive into this type of figure in our geometry journey for now.

Axiom 5 (SAS Congruence)**.** *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Axiom 6 (ASA Congruence)**.** *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

Axiom 7 (SSS Congruence)**.** *If two triangles have three sides congruent, then the triangles are congruent.*

Theorem 8 (Triangle Angle Sum)**.** *The sum of the three interior angles of any triangle is $180°$.*

Theorem 9 (Isosceles Base Angles)**.** *If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$. Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.*

Theorem 10 (Isosceles Median, Altitude, Angle Bisector)**.** *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*

Theorem 11 (Greater Sides Opposite Greater Angles)**.** *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Theorem 12 (Greater Angles Opposite Greater Sides)**.** *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

Theorem 13 (The triangle inequality)**.** *In $\triangle ABC$, we have $AB + BC > AC$.*

Theorem 14 (Perpendicular Bisector)**.** *The locus of points equidistant from a pair of points A, B is a line l which perpendicular to $\overline{AB}$ and goes through the midpoint of AB. This line is called the* perpendicular bisector *of $\overline{AB}$.*

Theorem 15 (Angle Bisector)**.** *For an angle ABC, the locus of points inside the angle which are equidistant from the two sides BA, BC is the ray $\overrightarrow{BD}$ which is the angle bisector of $\angle ABC$.*

Theorem 16. *In any triangle, the following results hold.*

1. *The perpendicular bisectors of three sides meet at a single point.*
2. *The three angle bisectors meet at a single point.*
3. *The three altitudes meet at a single point.*

Note: In original handouts, one part of this theorem was stated as Theorem 16, and other parts were given as HW problems.

3. Quadrilaterals

Definition 6 (Quadrilaterals)**.** A figure with four vertices, four corresponding sides, and four enclosed angles is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex A is opposite vertex C). Additionally, a quadrilateral is called

- a parallelogram, if both pairs of opposite sides are parallel.
- a rhombus, if all four sides have the same length.
- a trapezoid, if one pair of opposite sides are parallel. (these sides are called bases) and the other pair is not.
- a rectangle, if all its angles are equal (and thus equal to $90°$).

Theorem 17 (Parallelogram)**.** *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$

- *The intersection point M of diagonals AC and BD bisects each of them.*

Theorem 18 (Parallelogram 2)**.**

1. *Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = DC$, $AD = BC$. Then $ABCD$ is a parallelogram.*
2. *Let $ABCD$ be a quadrilateral such $AB = DC$, and $\overline{AB} \parallel \overline{DC}$. Then $ABCD$ is a parallelogram.*

Theorem 19 (Rhombus)**.** *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*

Definition 7 (Midline)**.** A midline of a triangle $\triangle ABC$ is a segment connecting the midpoints of any two sides; a midline of a trapezoid $ABCD$ with bases $\overline{AB}$, $\overline{CD}$ is the segment connecting the midpoints of the non-bases $\overline{AD}$, $\overline{BC}$.

Theorem 20 (Midline of a Triangle)**.** *If DE is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{AD} \parallel \overline{AC}$.*

4. Circles

Definition 8.

- A circle ω with center at O and radius r is the set of all points P such that $OP = r$.
- A radius is any line segment from O to a point A on ω
- A chord is any line segment between distinct points A, B on $\$\omega$
- A diameter is a chord that passes through O,
- A tangent line is a line that intersects the circle exactly once; if the intersection point is A, the tangent is said to be the tangent through A.

Theorem 21 (Tangent line)**.** *Let A be a point on circle ω centered at O, and m a line through A. Then m is tangent to ω if and only if $m \perp \overline{OA}$. Moreover, there is exactly one tangent to ω at A.*

Theorem 22 (Chord perpendicular bisector)**.** *Let $\overline{AB}$ be a chord of circle ω with center O. Then O lies on the perpendicular bisector of $\overline{AB}$. Moreover, if C is on $\overline{AB}$, then C bisects $\overline{AB}$ if and only if $\overline{OC} \perp \overline{AB}$.*

Theorem 23. *Let ω_1, ω_2 be circles with centers at points O_1, O_2 that intersect at points A, B. Then $\overline{AB} \perp \overline{O_1O_2}$.*

Theorem 24 (Tangent circles)**.** *Let ω_1, ω_2 be circles that are both tangent to line m at point A. Then these two circles have only one common point, A. Such circles are called* tangent.

Theorem 25 (Inscribed angle)**.** *Let A, B, C be on circle λ with center O. Then $\angle ACB = \frac{1}{2}\angle AOB$. The angle $\angle ACB$ is said to be inscribed in λ.*

5. Construction

Definition 9. Euclidea is a straightedge-compass construction game app, available on Google Play and the App Store; you can read about it at https://www.euclidea.xyz and https://twitter.com/euclidea_app

Definition 10. A straightedge is a tool that can construct a line through any two points; a compass is a tool that can construct a circle centered at a given point with radius congruent to a given line segment. A construction is a process whereby one constructs a geometric object by using tools to create other objects and then constructing any desired intersection points of the objects; a straightedge-compass construction is a construction using only straightedge and compass as tools.
In general, to provide a straightedge-compass construction of a figure, you may use any previous straightedge-compass constructions you have made as tools - for example, you may construct a parallel to any line through any point, since we have proven that a straightedge-compass construction of this is possible for any line and point.

So far, we have constructed the following items with straightedge and compass, which you may use in your constructions:

- A line segment $\overline{CD}$ congruent to $\overline{AB}$ given the points A, B, and C;
- An angle at a given ray congruent to another given angle;

- A line parallel to a line l and through a point P;
- A line perpendicular to a line l and through a point P;
- The perpendicular bisector of a line segment $\overline{AB}$;
- The center of a circle;
- The medians, altitudes, and angle bisectors of a triangle;
- A circle that passes through three given points.

6. Homework

1. Prove that a diameter is the longest chord of a circle (i.e., any chord that is not a diameter has smaller length than a diameter).

2. Must a quadrilateral with three congruent sides be a rhombus? Must a quadrilateral with two right angles be a rectangle?

3. Let E be a point on $\overline{AB}$ and points C, D on the same side of $\overline{AB}$ so that $\triangle ACE \cong \triangle EDB$. Prove that $\overline{DB} \parallel \overline{AB}$. What can we say about triangle $\triangle CDE$?

4. Suppose circles λ, ω with centers O, P respectively intersect at point A such that $m\angle OAP = x^\circ$ for some $x < 180$.
   (a) Prove that the tangents to λ, ω at A also intersect with an angle of measure x°.
   (b) Let B be the other point of intersection of λ, ω; prove that $\angle OBP \cong \angle OAP$.

5. Given a circle ω, a point A on ω, a line l that does not intersect ω, and a circle λ that is not inside ω, provide a straightedge-compass construction of:
   (a) The tangent to ω at A;
   (b) A tangent to ω parallel to l;
   (c) A line tangent to both ω and λ;
   (d) An isosceles trapezoid that passes through the centers of ω and λ (an isosceles trapezoid is a trapezoid whose non-base sides are congruent).

6. Given a triangle $\triangle ABC$,
   (a) Provide a straightedge-compass construction of a triangle $\triangle ABC'$ congruent to $\triangle ABC$;
   (b) Provide a straightedge-compass construction of a triangle $A'B'C'$ congruent to ABC such that corresponding sides of the two triangles are parallel;
   (c) Prove that $\overline{A'C}$ bisects $\overline{AC'}$ and vice-versa.

7. (a) Prove that a line cannot intersect a circle at three distinct points.
   (b) Prove that two circles cannot intersect at three distinct points.

8. Suppose λ, ω are circles centered at O, P, respectively, so that $\overline{OP}$ is a radius of both circlres. Then let μ be a circle centered at O with radius $\frac{1}{2}OP$ which intersects ω at A, and let ν be a circle with radius $\frac{1}{2}OP$ centered at A.
   (a) Prove that λ is tangent to ν.
   (b) Let B be the point of intersection of λ and ν, and let M be the midpoint of $\overline{OP}$; prove that $\triangle OBM$ is isosceles.
   (c) Let $\overline{OA}$ intersect the perpendicular bisector of $\overline{OP}$ at N. Prove that $ON = 2OP$.
   (d) Prove that $\overleftrightarrow{AM}$ intersects $\overleftrightarrow{NP}$ at a point on ω.