MATH 8: ASSIGNMENT 15

JANUARY 19, 2020

1. Constructions with ruler and compass

Now that we know when two geometric objects are the same (via congruence), it makes sense to ask if we can produce figures with specific properties of interest — for example, if we can reproduce a given angle somewhere else so that the resulting angle is congruent to the original. Traditionally, such constructions are done using straight-edge and compass: the straight-edge tool constructs lines and the compass tool constructs circles. More precisely, it means that we allow the following basic operations:

- Draw (construct) a line through two given or previously constructed distinct points. (Recall that by axiom 1, such a line is unique).
- Draw (construct) a circle with center at previously constructed point O and with radius equal to distance between two previously constructed points B, C
- Construct the intersections point(s) of two previusly constructed lines, circles, or a circle and a line

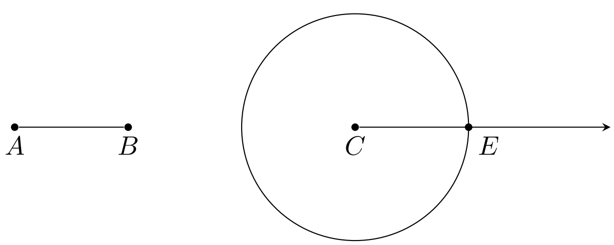
All other constructions (e.g., draw a line parallel to a given one) must be done using these elementary constructions only!!

Constructions of this form have been famous since mathematics in ancient Greece.

Here are some examples of constructions:

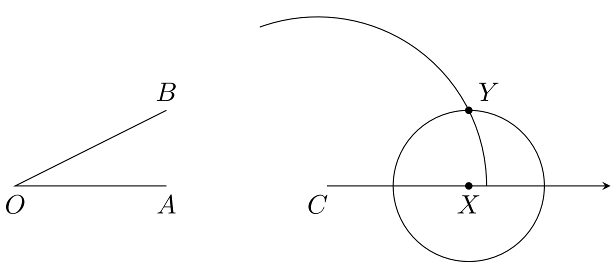
Example 1. Given any line segment $\overline{AB}$ and ray $\overrightarrow{CD}$, one can construct a point E on $\overrightarrow{CD}$ such that $\overline{CE} \cong \overline{AB}$.

Construction. Construct a circle centered at C with radius AB. Then this circle will intersect $\overrightarrow{CD}$ at the desired point E. □



Example 2. Given angle $\angle AOB$ and ray $\overrightarrow{CD}$, one can construct an angle around $\overrightarrow{CD}$ that is congruent to $\angle AOB$.

Construction. First construct point X on $\overrightarrow{CD}$ such that $CX \cong OA$. Then, construct a circle of radius OB centered at C and a circle of radius AB centered at X. Let Y be the intersection of these circles; then $\triangle XCY \cong \triangle AOB$ by SSS and hence $\angle XCY \cong \angle AOB$. □



2. Perpendicualr Bisector

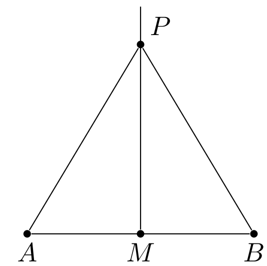
Consider any property of points on the plane — for example, the property that a point P is a distance exactly r from a given point O. The set of all points P for which this property holds true is called the locus of points satisfying this property. As we have seen above, the locus of points that are a distance r from a point O is called a circle (specifically, a circle of radius r centered at O).

Now consider we are given two points A, B. If a point P is an equal distance from A, B (i.e., if $\overline{PA} \cong \overline{PB}$) then we say P is equidistant from points A, B.

Theorem 14. *The locus of points equidistant from a pair of points A, B is a line l which perpendicular to $\overline{AB}$ and goes through the midpoint of AB. This line is called the perpendicular bisector of $\overline{AB}$.*

Proof. Let M be the midpoint of $\overline{AB}$, and let l be the line through M which is perpendicular to AB (such a line exists and is unique — prove it!). We need to prove that for any point P,

$$(AP \cong BP) \iff P \in l$$



1. Assume that $AP \cong BP$. Then triangle APB is isosceles; by Theorem 10 from last week, it implies that $PM \perp AB$. Thus, PM must coincide with l, i.e. $P \in l$. Therefore, we have proved implication one way: if $AP \cong BP$, then $P \in l$.

2. Conversley, assume $P \in l$. Then $m\angle AMP = m\angle BMP = 90°$; thus, triangles $\triangle AMP$ and $\triangle BMP$ are congruent by SAS, and therefore $AP \cong BP$.

□

The notion of a geometric locus can be used to produce fascinating figures and objects; some of the most famous of these are known collectively as the conic sections (circle, ellipse, parabola, hyperbola, pair of lines.).

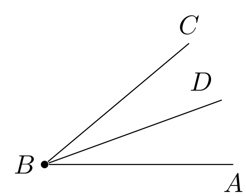
3. Median, Altitude, Angle Bisector

Last week we defined three special lines that can be constructed from any vertex in any triangle; each line goes from a vertex of the triangle to the line containing the triangle's opposite side (altitudes may sometimes land on the opposite side outside of the triangle).
Given a triangle $\triangle ABC$,

- The altitude from A is the line through A perpendicular to $\overleftrightarrow{BC}$;
- The median from A is the line from A to the midpoint D of $\overline{BC}$;
- The angle bisector from A is the line $\overleftrightarrow{AE}$ such that $\angle BAE \cong \angle CAE$. Here we let E denote the intersection of the angle bisector with $\overline{BC}$.

The following result is an analog of theorem 14. For a point P and a line l, we define the distance from P to l to be the length of the perpendicualr dropped from P to l (see problem 1 in the HW). We say that point P is equidistant from two lines l, m if the distance from P to l is equal to the distance from P to m.

Theorem 15. *For an angle ABC, the locus of points inside the angle which are equidistant from the two sides BA, BC is the ray $\overrightarrow{BD}$ which is the angle bisector of $\angle ABC$.*



Proof of this theorem is given as a homework.

4. Homework

1. Let P be a point not on line l, and $A \in l$ be the base of perpendicular from P to l: $AP \perp l$. Prove that for any other point B on l, $PB > PA$ ("perpendicular is the shortest distance"). Note: you can not use Pythagorean theorem as we have not proved it yet; instead, try using Theorem 11.

2. Suppose $\triangle ABC$ is an isosceles right triangle with right angle $\angle A$. Let $\triangle BCD$ be an isosceles right triangle with right angle $\angle B$ such that $\overline{AD}$ intersects $\overline{BC}$. Prove that $\overline{AE}$ is an altitude of $\triangle ABC$.

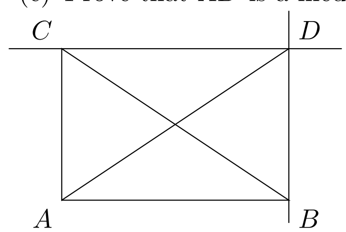
3. Let $\triangle ABC$ be a right triangle with right angle $\angle A$, and let D be a point on $\overline{BC}$. Prove that $\triangle ADB$ is isosceles if and only if $\triangle ADC$ is isosceles.

4. Let $\triangle ABC$ be a right triangle with right angle $\angle A$, and let D be the intersection of the line parallel to $\overline{AB}$ through C with the line parallel to $\overline{AC}$ through B.

(a) Prove $\triangle ABC \cong \triangle DCB$

(b) Prove $\triangle ABC \cong \triangle BDA$

(c) Prove that $\overline{AD}$ is a median of $\triangle ABC$.



5. Let $\triangle ABC$ be a right triangle with right angle $\angle A$, and let D be the midpoint of $\overline{BC}$. Prove that $AD = \frac{1}{2}BC$.

6. Let l_1, l_2 be the perpendicular bisectors of side AB and BC respectively of $\triangle ABC$, and let F be the intersection point of l_1 and l_2. Prove that then F also lies on the perpendicular bisector of the side BC. [Hint: use Theorem 14.]

7. Prove Theorem 15.

8. Let the angle bisectors from B and C in the triangle $\triangle ABC$ intersect each other at point F. Prove that $\overleftrightarrow{AF}$ is the third angle bisector of $\triangle ABC$. [Hint: use Theorem 15]

9. Let $\triangle ABC$ be isosceles with $\overline{AB} \cong \overline{AC}$. Let the altitudes from B and C intersect their opposite legs at the points D and E respectively, and let $\overline{BD}$, $\overline{CE}$ intersect at F.

(a) Prove $\angle EBF \cong \angle DCF$

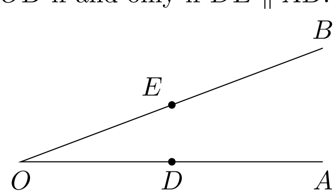
(b) Prove $\triangle DBC \cong \triangle ECB$

(c) Prove $\triangle DCF \cong \triangle EBF$

(d) Prove $\triangle AEF \cong \triangle ADF$

(e) Prove that $\overleftrightarrow{AF}$ is the third altitude of $\triangle ABC$.

10. Given line segments $\overline{OA}$ and $\overline{OB}$ and midpoint D of $\overline{OA}$, prove that a point E on $\overline{OB}$ is the midpoint of $\overline{OB}$ if and only if $\overline{DE} \parallel \overline{AB}$.



11. Given triangle $\triangle ABC$, let D, E be the midpoints of sides $\overline{AB}$, $\overline{AC}$ respectively. Prove that $\overline{DE} \parallel \overline{BC}$ and $DE = \frac{1}{2}BC$. (The line segment $\overline{DE}$ is called the midline of the triangle from A.)