

**MATH 8**  
**ASSIGNMENT 11: LOGIC REVIEW**  
DEC 8, 2019

STATEMENTS AND QUANTIFIERS

1. Given statements  $a$  and  $b$ , if I know that  $\neg(a \wedge b)$  is true and I know that  $a$  is true, what can I conclude about  $b$ ?
2. Recall that  $\mathbb{Z}$  denotes the set of integers. Prove or disprove the following statement:  $\forall p \in \mathbb{Z}$  ( $p$  is prime  $\implies p + 1$  is not a power of 2)
3. Given some object  $x$ , let  $w(x)$  indicate the statement “ $x$  has wheels”, and  $d(x)$  indicate “ $x$  can drive”. Let  $L$  indicate the universe of objects that exist solely on land. So, for example, let my car be called  $c_9$ . Then  $w(c_9)$  is true and  $d(c_9)$  is true.
  - (a) Consider the following claim: “any object that exists on land must have wheels in order to drive”. How can we represent this as a logical statement?
  - (b) Suppose I find a wheel factory, which I will call  $f$ . This factory has wheels but cannot drive. Is  $w(f)$  true? What about  $d(f)$ ?
  - (c) Given the existence of the factory  $f$ , prove or disprove the following statement:  $\forall x \in L(w(x) \implies d(x))$ .

LOGICAL EQUIVALENCE

1. Prove that a positive integer is odd if and only if it can be expressed as a difference of consecutive squares. (Here, *consecutive squares* means the squares of two positive integers  $y, y + 1$ ).
2.
  - (a) Given that the product of zero with any integer is zero, and the product of any two nonzero integers is nonzero, prove the following:  $\forall x \in \mathbb{Z}(\forall y \in \mathbb{Z}((xy = 0) \iff (x = 0 \vee y = 0)))$
  - (b) Given some integer  $a$ , prove that  $\forall x \in \mathbb{Z}((x - a)^2 = 0 \iff x = a)$
  - (c) Prove that  $\forall x \in \mathbb{Z}(x^2 - 4x + 4 = 0 \iff x = 2)$
3. Suppose I am producing haute-couture fine art on Microsoft Paint, and I begin by opening a blank, white artwork and drawing two black lines. I extend the lines as far as possible so that they intersect the edge of the painting. I then use the color-fill tool, but due to time and dignity constraints, I am only allowed to use it three times. If my goal is to render as much of my artwork as possible into color, prove that there will be a white region left over if and only if the lines I drew intersect each other.
4. Suppose I have three logical statements  $a, b, c$ , and I want to prove they are all equivalent. That is, I wish to prove  $a \iff b, b \iff c$ , and  $a \iff c$ .
  - (a) Show that if  $a \implies b$  and  $b \implies c$  then  $a \implies c$ .
  - (b) Suppose we manage to prove  $a \implies b, b \implies c$ , and  $c \implies a$ . Is this enough to prove that  $a \iff c$ ? [Hint: you can conclude  $a \iff c$  from  $a \implies c$  and  $c \implies a$ .]

(c) Is  $a \implies b$ ,  $b \implies c$ , and  $c \implies a$  enough to prove  $a \iff b$  and  $b \iff c$ ?

PROOF BY CONTRADICTION

1. Recall that the product of any two positive real numbers is positive. Given real numbers  $x$ ,  $y$ ,  $z$ , such that  $y > z$  and  $xy < xz$ , prove that  $x < 0$ . [
2. Recall that an integer  $x$  is *even* if  $x = 2k$  for some integer  $k$ , and  $x$  is *odd* if  $x = 2k + 1$  for some integer  $k$ . Given two positive integers  $m$ ,  $n$  such that  $mn$  is even, prove that  $m$  is even or  $n$  is even. (You may assume that any integer that is not even must be odd.)
3. Given that no positive integer is a factor of 1, prove the following statements:
  - (a) If  $x$  is even, then  $x + 1$  is not divisible by 2.
  - (b) If  $x$  is divisible by 5, then  $x + 1$  is not divisible by 5.
  - (c) If  $x + 1$  is divisible by 5, then  $x$  is not divisible by 5.
  - (d) For any integer  $x > 1$ ,  $x$  and  $x + 1$  have no common factors. (A *common factor* is a positive integer  $k$  that divides both of the integers in question.)