MATH 8
ASSIGNMENT 2: BINOMIAL FORMULA

SEPT. 22, 2019

Pascal triangle

Recall the numbers

$$ {}_nC_k = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!k!} \tag{1}$$

These numbers appear in many problems:

$$\begin{aligned} {}_nC_k &= \text{ The number of ways to choose } k \text{ items out of } n \text{ if the order does not matter} \\ &= \text{The number of words that can be written using } k \text{ zeros and } n-k \text{ ones} \end{aligned}$$

These numbers have the following useful property:

$$ {}_{n+1}C_k = {}_nC_{k-1} + {}_nC_k \tag{2}$$

For example, ${}_7C_4 = {}_6C_3 + {}_6C_4$ (check!).

These can be argued in two ways: either from definition (discussed in class) or from formula (1) (see Problem 7 below).

Thus, if we arrange these numbers in a triangle, so that the k-th entry in n-th row is ${}_nC_k$ (note that both n and k are counted from 0, not from 1: for example, $\binom{2}{1} = 2$), then every entry in this triangle is obtained as the sum of two entries above it.

This triangle is called the Pascal triangle:

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&2&&1&&&&&& \\
&&&&&1&&3&&3&&1&&&&& \\
&&&&1&&4&&6&&4&&1&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

Binomial formula

These numbers have one more important application:

$$(a+b)^n = {}_nC_0 a^n + {}_nC_1 a^{n-1} b^1 + \cdots + {}_nC_n b^n \tag{3}$$

The general term in this formula looks like ${}_nC_k \cdot a^{n-k} b^k$. For example, for $n = 3$ we get

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

(compare with the 3rd row of Pascal triangle)

This formula is called the **binomial formula**; we discussed its proof today.

Problems

1. Use the binomial formula to expand the following expressions:

(a) $(x-y)^3$
(b) $(a+3b)^3$
(c) $(2x+y)^5$
(d) $(x+2y)^5$

2. Deduce that the Pascal triangle is symmetric, i.e. ${}_nC_k = {}_nC_{n-k}$ in two ways:

(a) Use the binomial formula for $(x+y)^n$ and $(y+x)^n$.
(b) Using formula (1).

3. Use the binomial formula to compute

(a) Sum of all numbers in the nth row of Pascal triangle. [Hint: take $a=b=1$ in the binomial formula.]
(b) Alternating sum of all numbers in the nth row of Pascal triangle: ${}_nC_0 - {}_nC_1 + {}_nC_2 - {}_nC_3 \dots$

4. Let p be prime.

(a) Show that each of the binomial coefficients ${}_pC_k$, $1 \le k \le p-1$, is divisible by p.
(b) Show that if a, b are integer, then $(a+b)^p - a^p - b^p$ is divisible by p.

5. Show that $(1+\sqrt{2})^{12} + (1-\sqrt{2})^{12}$ is integer.

6. A drunk men is moving along a street. Every 10 seconds he randomly moves either one step up the street (say, due north) or down the street (south). What is the probability that after 10 minutes (that is, after making 60 steps) he will come back to the place he started with? move exactly one step north? exactly two steps north? will be within 10 steps from his starting position?

7. Deduce formula (2) from (1):

(a) Explain why

$$\frac{6!}{3!3!} + \frac{6!}{2!4!} = \frac{7!}{3!4!}$$

Can you do without explicitly computing all three numbers?

(b) Explain why

$$\frac{50!}{12!38!} + \frac{50!}{11!39!} = \frac{51!}{12!39!}$$

(hint: $12! = 11! \cdot 12$, so one can rewrite $\frac{50!}{12!38!} = \frac{50!}{11!38!} \cdot \frac{1}{12}$. Similarly we can rewrite all other terms...)

(c) Repeat the same arguments (with necessary changes) to prove

$${}_{n+1}C_k = {}_nC_{k-1} + {}_nC_k$$