MATH 8: HANDOUT 14
GEOMETRY III: PERPENDICULAR BISECTOR, ANGLE BISECTOR

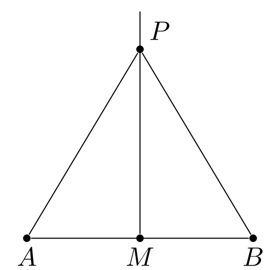
1. Perpendicular Bisector

Consider any property of points on the plane — for example, the property that a point P is a distance exactly r from a given point O. The set of all points P for which this property holds true is called the locus of points satisfying this property. As we have seen above, the locus of points that are a distance r from a point O is called a circle (specifically, a circle of radius r centered at O).
Now consider we are given two points A, B. If a point P is an equal distance from A, B (i.e., if $\overline{PA} \cong \overline{PB}$) then we say P is equidistant from points A, B.

Theorem 14. *The locus of points equidistant from a pair of points A, B is a line l which perpendicular to $\overline{AB}$ and goes through the midpoint of AB. This line is called the perpendicular bisector of $\overline{AB}$.*

Proof. Let M be the midpoint of $\overline{AB}$, and let l be the line through M which is perpendicular to AB. We need to prove that for any point P,

$$(AP \cong BP) \iff P \in l$$



1. Assume that $AP \cong BP$. Then triangle APB is isosceles; by Theorem 10 from last week, it implies that $PM \perp AB$. Thus, PM must coincide with l, i.e. $P \in l$. Therefore, we have proved implication one way: if $AP \cong BP$, then $P \in l$.
2. Conversly, assume $P \in l$. Then $m\angle AMP = m\angle BMP = 90°$; thus, triangles $\triangle AMP$ and $\triangle BMP$ are congruent by SAS, and therefore $AP \cong BP$.

□

Theorem 15. *In a triangle $\triangle ABC$, the perpendicular bisectors of the 3 sides intersect at a single point. This point is the center of a circle circumscribed about the triangle (i.e., such that all three vertices of the triangle are on the circle).*

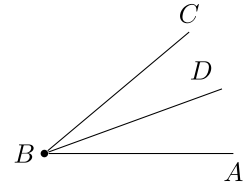
2. Median, Altitude, Angle Bisector

Last week we defined three special lines that can be constructed from any vertex in any triangle; each line goes from a vertex of the triangle to the line containing the triangle's opposite side (altitudes may sometimes land on the opposite side outside of the triangle).
Given a triangle $\triangle ABC$,

- The altitude from A is the line through A perpendicular to $\overleftrightarrow{BC}$;
- The median from A is the line from A to the midpoint D of $\overline{BC}$;
- The angle bisector from A is the line $\overleftrightarrow{AE}$ such that $\angle BAE \cong \angle CAE$. Here we let E denote the intersection of the angle bisector with $\overline{BC}$.

The following result is an analog of theorem 14. For a point P and a line l, we define the distance from P to l to be the length of the perpendicular dropped from P to l (see problem 1 in the HW). We say that point P is equidistant from two lines l, m if the distance from P to l is equal to the distance from P to m.

Theorem 16. *For an angle ABC, the locus of points inside the angle which are equidistant from the two sides BA, BC is the ray $\overrightarrow{BD}$ which is the angle bisector of $\angle ABC$.*



Proof of this theorem was discussed in class.

Homework

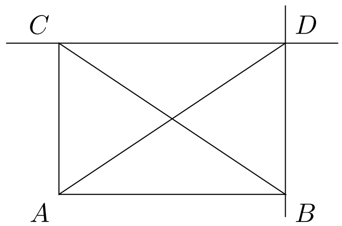
1. Let P be a point not on line l, and $A \in l$ be the base of perpendicular from P to l: $AP \perp l$. Prove that for any other point B on l, $PB > PA$ ("perpendicular is the shortest distance"). Note: you can not use Pythagorean theorem as we have not proved it yet; instead, try using Theorem 11 (opposite the larger angle there is a longer side).

2. Let $\triangle ABC$ be a right triangle with right angle $\angle A$, and let D be the intersection of the line parallel to $\overline{AB}$ through C with the line parallel to $\overline{AC}$ through B.

(a) Prove $\triangle ABC \cong \triangle DCB$

(b) Prove $\triangle ABC \cong \triangle BDA$

(c) Prove that $\overline{AD}$ is a median of $\triangle ABC$.



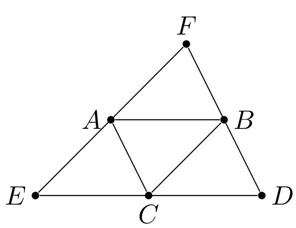
3. Let $\triangle ABC$ be a right triangle with right angle $\angle A$, and let D be the midpoint of $\overline{BC}$. Prove that $AD = \frac{1}{2}BC$.

4. Let l_1, l_2 be the perpendicular bisectors of side AB and BC respectively of $\triangle ABC$, and let F be the intersection point of l_1 and l_2. Prove that then F also lies on the perpendicular bisector of the side BC. [Hint: use Theorem 14.]

5. Prove Theorem 15.

6. Let the angle bisectors from B and C in the triangle $\triangle ABC$ intersect each other at point F. Prove that $\overleftrightarrow{AF}$ is the third angle bisector of $\triangle ABC$. [Hint: use Theorem 16]

7. Given triangle $\triangle ABC$, draw through each vertex a line parallel to the opposite side. Denote the vertices of the resulting triangle by D, E, F, as shown in the figure below.



(a) Prove that $\triangle ABC \cong \triangle BAF$ (pay attention to the order of vertices). Similarly one proves that all four small triangles in the picture are congruent.

(b) Prove that $\overline{AB} \parallel \overline{ED}$ and $AB = \frac{1}{2}ED$.

(c) Prove that perpendicular bisectors of sides of $\triangle DEF$ are altitudes of $\triangle ABC$.

(d) Show that in any triangle, the three altitudes meet at a single point.