MATH 8

HANDOUT 11: LOGIC REVIEW

NOTATION REMINDER

Logical operations: $\vee$ or; $\wedge$ and; $\neg$ not; $\implies$ implies (same as "if... then...");
Quantifiers: $\forall x \in A : \dots$ for any x in set A, ...
$\exists x \in A : \dots$ there exists an x in set A such that ...
Common notation for sets:
$\mathbb{R}$: set of real numbers
$\mathbb{N}$: set of positive integers (1,2,...)
$\mathbb{Z}$: set of integers (including zero and negatives)

PROBLEMS

When doing these problems, you can use the following standard facts about sets $\mathbb{R}$ and $\mathbb{Z}$:

- Laws for addition and multiplication: commutativity, associativity, distributivity
- Rules for 0 and 1: $0 + x = x$, $1 \cdot x = x$, $0 \cdot x = 0$.
- Rules for negatives: for any x, there is a unique number $-x$ such that $x + (-x) = 0$, and $-(-x) = x$.
- (For set $\mathbb{R}$ only): any nonzero $x \in \mathbb{R}$ has an inverse: there exists a unique number x^{-1} such that $x \cdot x^{-1} = 1$.

1. Given statements A and B, if I know that $\neg(A \wedge B)$ is true and I know that A is true, what can I conclude about B?

2. An integer number a is called even if $\exists n \in \mathbb{Z} : a = 2n$. A number a is called odd if $\exists n \in \mathbb{Z} : a = 2n + 1$. You can use without proof the fact that every integer is either even or odd, but not both.
  (a) Prove that if a number is even, then its square is also even
  (b) Prove that if integers a, b are even, then $a + b$ is also even.
  (c) Prove that an integer number a is odd if and only if a^2 is odd
  (d) Show that if mn is even, then m is even or n is even.

3. For integer numbers a, b, we say that a divides b, or that b is divisible by a (notation: $a|b$) is there exists an integer n such that $b = na$.
  (a) Prove that if $a|b$ and $a|c$, then $a|(b + c)$.
  (b) Prove that if $a|b$ but a doesn't divide c, then a doesn't divide $b + c$.

4. Prove that a positive integer a is odd if and only if it can be expressed as a difference of consecutive squares. (Here, *consecutive squares* means the squares of two positive integers k, $k + 1$).

5. It is known that integer 1 is not divisible by any positive integer other than 1. Use it to prove:
  (a) If integer n is even, then $n + 1$ is not divisible by 2.
  (b) If integer n is divisible by 5, then $n + 1$ is not divisible by 5.
  (c) If $n + 1$ is divisible by 5, then n is not divisible by 5.
  (d) For any integer $n > 1$, n and $n + 1$ have no common factors other than 1. (A *common factor* is a positive integer k that divides both of the integers in question.)

6. Prove or disprove the following statement: $\forall p \in \mathbb{Z}$ (p is prime $\implies$ $p + 1$ is not a power of 2)

7. (a) Prove that if $x, y \in \mathbb{R}$ are such that $xy = 0$, then $x = 0 \vee y = 0$. [Hint: x has an inverse.]
  (b) Prove (using nothing but the basic facts about reals given above) that $x^2 - 5x + 6 = 0$ if and only if $x = 2$ or $x = 3$. Here x is a real number.

8. Recall that the product of any two positive real numbers is positive. Given real numbers x, y, z, such that $y > z$ and $xy < xz$, prove that $x < 0$.

9. The standard card deck has 52 cards (4 suits, each with 13 values, from 2 to ace). You have drawn 2 cards, which happened to be king of spades and 9 of hearts.

You now draw 3 more cards. What is the probability that your 5-card hand will contain the following combinations:
  (a) Four of a kind: four cards of the same value
  (b) Three of a kind: three card of the same value (but not 4 cards of the same value)
  (c) Two kings and two nines (but no three of a kind)
  (d) Three kings and two nines.