MATH 8
HANDOUT 7: LOGIC III — INTRODUCTION TO SYMBOLS AND FORMULAS

Last week we looked at a set of rules for a formal system of objects, and attempted to discover some of the nature of this universe, using only the power of logic for our proofs. Today we will get more comfortable with a few concepts, and a few symbols, used in mathematical logic. In particular, we will use the symbols $\wedge$, $\vee$, $\implies$, $\neg$, and $\iff$.

Consider you are inside a room with some lights and you spot a switch on the wall. You flip the switch and take note of whether the light turns on. For convenience, we will denote the switch being in the on position as S and the lights being on as L. In particular,

- If the switch is not in the on position, then it is in the off position, $\neg S$, read **"not S"**. If the lights are off, then the lights are not on: $\neg L$.
- If the light turns on, then the switch seems to be connected to the light. Therefore $S \implies L$, read **"S implies L"**.
- When the switch and the lights are both on, we have $S \wedge L$, read **"S and L"**.
  Remember that the $\wedge$ condition requires *both* logical statements to be true. So $S \wedge L$ is false if S is false or L is false, even if the other is true.
- What if the light didn't turn on? If we flip the switch and the lights remain off, we have $S \wedge \neg L$, **"S and not L"**.
- We then know that turning the switch on does not result in the lights turning on, therefore $\neg(S \implies L)$, **"not (S implies L)"** or **"S does not imply L"**. Note that while it is possible to negate S or L, it is also possible to negate longer statements like $S \implies L$.
- $S \vee L$, read **"S or L"** indicates that the switch is on or the lights are on.
  Note that this includes the possibility where both are true (unlike in English language, where *or* is often *exclusive*).
- Now there is one more concept we should be familiar with: let's say the light switch corresponds perfectly to the light — i.e., when the switch is on the light is on, and when the switch is off the light is off. Then S is true if and only if L is true, which we write as $S \iff L$, **"S if and only if L"**.

Classwork and Homework

1. (a) Simplify $A \vee (A \wedge B)$
(b) Simplify $A \wedge (A \vee B)$

2. (a) Simplify $\neg(\neg A)$
(b) $\neg(\neg(\neg A)$

3. (a) Simplify $(A \implies B) \wedge A$
(b) Simplify $(A \implies B) \wedge \neg A$

4. The concept $\iff$ is sometimes called *logical equivalence*.
(a) Given statements A and B, prove that $(A \implies B) \wedge (B \implies A)$ results in $A \iff B$.
(b) Now consider A and B are conjectures we are trying to prove something about. If we prove the following two statements:

$$A \implies B$$
$$B \implies A$$

Have we proved that $A \iff B$?

5. The concept $\iff$ is also called *if and only if*.
(a) Given statements A, B, write down a logical formula for "if A, then B".
(b) Write down a logical formula for "A is true only if B is true".
(c) Combine your answers for the above two parts and simplify the resulting expression.

6. Given three logical statements A, B, C, prove that $(A \wedge B) \wedge C$ is true if and only if $A \wedge (B \wedge C)$.

7. Given statements A, B,
(a) When is $A \wedge B$ false?
(b) Write the logical formula for $\neg(A \wedge B)$.
(c) When is $A \vee B$ false?
(d) Write the logical formula for $\neg(A \vee B)$.

8. Given statements A, B, C, prove that $(A \implies B) \wedge (B \implies C) \wedge (C \implies A)$ implies both $A \iff B$ and $A \iff C$.

9. Given a statement A, $A \wedge \neg A$ can't possibly be true (it is a contradiction!). Recall Rule 2 from last homework - which states that, given objects x and y, $(x < y) \implies \neg(y < x)$.
(a) Given some object x, since Rule 2 must be true for any objects x, y, substitute x for y and write down the result.
(b) Let A be the logical statement $x < x$. Rewrite the above substituted statement in terms of A.
(c) Deduce that, using only Rule 2, $x < x$ is a contradiction for an arbitrary object x. Conclude that Rule 2 $\implies$ Rule 1.

10. Imagine some objects that follow the rules of the last homework.
(a) Imagine that you have two collections of objects, each of which is linked somehow to every object in the other group, but no two objects in the same group are linked. Draw or describe a setup of links that makes this possible.
(b) Consider now having three collections of objects, each of which is linked somehow to every object in both other groups, but again, no two objects in the same group are linked. Draw or describe a setup of links that makes this possible.

11. Simplify $\binom{n}{k} + \binom{n}{k+1}$.

12. Which coefficient in the n^{th} row of Pascal's Triangle is the largest? Can you prove why?