MATH 8
HANDOUT 6: LOGIC LAWS AND PROOFS

REVIEW: LOGICAL VARIABLES AND TRUTH TABLES

Logical variables: take value True (T) or False (F).

Basic logic operations:

- **NOT:** (for example, NOT A): true if A is false, and false if A is true. Commonly denoted by $\neg A$ or (in computer science) $!A$.
- **AND:** (for example A AND B): true if both A, B are true, and false otherwise . Commonly denoted by $A \wedge B$
- **OR:** (for example A OR B): true if at least one of A, B is true, and false otherwise. Sometimes also called "inclusive or" to distinguish it from the "exclusive or" described below. Commonly denoted by $A \vee B$.
- **IF:** (as in "if A, then B; written $A \implies B$): if A is false, automatically true; if A is true, it is true only when B is true

As in usual algebra, logic operations can be combined, e.g. $(A \vee B) \wedge C$.

Truth tables

If we have a logical formula involving variables $A, B, C, \dots$, we can make a table listing, for every possible combination of values of $A, B, \dots$, the value of our formula. For example, the following is the truth tables for OR and IF:

A	B	A OR B
T	T	T
T	F	T
F	T	T
F	F	F

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

LOGIC LAWS

We can combine logic operations, creating more complicated expressions such as $A \wedge (B \vee C)$. As in arithmetic, these operations satisfy some laws: for example $A \vee B$ is the same as $B \vee A$. Here, "the same" means "for all values of A, B, these two expressions give the same answer"; it is usually denoted by $\iff$. Here are two other laws (see problem 5):

$$\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$$
$$(A \implies B) \iff ((\neg B) \implies (\neg A))$$

Truth tables provide the easiest way to prove complicated logical rules: if we want to prove that two formulas are equivalent (i.e., always give the same answer), make a truth table for each of them, and if the tables coincide, they are equivalent.

PROOFS

What is a proof?

Common answer is: a sequence of statements, starting with given ones and ending in a statement which we want to prove, and such that each statement in the sequence logically follows from the previous.

What exactly "logically follows" means?

In the simple case when all our statements can be written as combinations of the same elementary statements (which we can denote by letters $A, B, \dots$), using logical operations, it means the following:

> For any combinations of values of letters $A, B, \dots$ which makes the previous statements true, the next statement is also true.

Thus, it can be checked simply by a truth table. E.g., statement $\neg A$ logically follows from $A \implies B$ and $\neg B$: in all cases when $A \implies B$ and $\neg B$ are true, $\neg A$ is also true, as is easy to check.

However, usually instead of writing truth tables, people use some simple logical laws repeatedly. Here are some commonly used laws:

- Given $A \implies B$ and A, we can conclude B.
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! it only means that **if** A is true, then so is C.]
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$

These laws have some historical Latin names, such as *Modus Ponens*, but you do not need to know that :)

HOMEWORK

1. Check whether $A \implies B$ and $B \implies A$ are equivalent, by writing the truth table for each of them.

2. Check that $A \implies B$ is equivalent to $(\neg A) \vee B$ (thus, "if you do not clean up your room, you will be punished" and "clean up your room, or you will be punished" are the same).

3. A teacher tell the student "If you do not take the final exam, you get an F". Does it mean that

(a) If the student does take the final exam, he will not get an F

(b) If the student does not get an F, it means he must have taken the final exam.

4. Write the truth table for each of the following formulas. Are they equivalent (i.e., do they always give the same value)?

(a) $(A \vee B) \wedge (A \wedge C)$

(b) $A \vee (B \wedge C)$.

5. Use the truth tables to prove *De Morgan's laws*

$$\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$$
$$\neg(A \vee B) \iff (\neg A) \wedge (\neg B)$$

6. There is a room in your building that is controlled by an economic light switch: when switched off, the lights are off, and when switched on, the lights are either on or off depending on whether anyone has moved in the room recently (as picked up by the light's motion detector).

(a) You see that the light is on; can you conclude that the switch is on?

(b) You see that the light is off; can you conclude that the switch is off?

(c) Draw out a truth table for the light and the switch: for each combination of the light and switch being on or off, mark whether that combination is possible. Then compare this to the truth table of (light is on $\implies$ switch is on)

7. If today is Thursday, then Jane's class has library day. If Jane's class has library day, then Jane will bring home new library books. Jane brought no new library books. Therefore,...

Can you guess what would be the natural conclusion from these 3 statements? Can you prove it using some laws of logic? It might help to write each of them as combination of elementary statements e.g. T for "today is Tuesday", L for "Jane's class has a library day", etc.

8. On the island of knights and knaves, there are two kinds of people: Knights, who always tell the truth, and Knaves, who always lie. Unfortunately, there is no easy way of knowing whether a person you meet is a knight or a knave...

You meet two people on this island, Bart and Ted. Bart claims, "I and Ted are both knights or both knaves." Ted tells you, "Bart would tell you that I am a knave." So who is a knight and who is a knave?

9. You probably know Lewis Carroll as the author of *Alice in Wonderland* and other books. What you might not know is that he was also a mathematician very much interested in logic, and had invented a number of logic puzzles. Here is one of them:

You are given 3 statements.

(a) All babies are illogical.

(b) Nobody is despised who can manage a crocodile.

(c) Illogical persons are despised.

Can you guess what would be the natural conclusion from these 3 statements? Can you prove it using some laws of logic?

It might help to write each of them as combination of elementary statements about a given person, e.g. B for "this person is a baby", I for "this person is illiogical", etc.

10. Here is another one of Lewis Carrol's puzzles.

All hummingbirds are richly colored..

No large birds live on honey.

Birds that do not live on honey are dull in color.

Therefore,. . .