

**MATH 8**  
**HANDOUT 3: FORMULA FOR  $\binom{n}{k}$**

MAIN FORMULAS OF COMBINATORICS

Recall the numbers  $\binom{n}{k}$  from the Pascal triangle:

- $\binom{n}{k}$  = The number of paths on the chessboard going  $k$  units down and  $n - k$  to the right  
= The number of words that can be written using  $k$  zeros and  $n - k$  ones  
= The number of ways to choose  $k$  items out of  $n$  if the **order does not matter**

It turns out that there is an explicit formula for them:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!k!}$$

Thus, we now have a full list of all the main formulas of combinatorics:

- The number of ways to order  $k$  items is

$$k! = k(k-1)\cdots 2 \cdot 1$$

- The number of ways to choose  $k$  items out of  $n$  if the **order matters** is

$${}_nP_k = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- The number of ways to choose  $k$  items out of  $n$  if the **order does not matter** is

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!k!}$$

PROBLEMS

In all the problems, you can write your answer as a combination of factorials and  $\binom{n}{k}$  – you do not have to do the computations. And, as usual, please write your reasoning, not just the answers!

1. A senior class in a high school, consisting of 120 students, wants to choose a class president, vice-president, and 3 steering committee members. How many ways are there for them to do this?
2. A party of 5 men sits at a bar, which offers 17 different cocktails.
  - (a) If each man wants to order a cocktail, how many possible combinations are there?
  - (b) If each man wants to order a cocktail so that all of them have different cocktails, how many possible combinations are there?
  - (c) How many ways are there to choose 5 cocktails out of 17 (all 5 different, but the order doesn't matter)?
3. Andrew has 7 pieces of candy, and Tim has 9 (all different). They want to trade 5 pieces of candy. How many ways are there for them to do it?
4. If you have 5 lines on the plane so that no two are parallel and there are no triple intersection points, how many triangles do they form? What if there are  $n$  lines?

5. In one of the lotteries run by New York State, “Sweet Million”, they randomly choose 6 numbers out of numbers 1–40. If you guess all 6 correctly (order does not matter), you win \$1,000,000. [There are also smaller prizes for guessing 5 out of 6, etc., but let us ignore them for now.]
  - (a) How many ways are there to choose 6 numbers out of 40?
  - (b) What are your chances of winning?
  - (c) If a lottery ticket cost \$1, how much money does New York State make for each ticket sold (on average)?
  - \*(d) If you choose 6 numbers out of 40 at random, what are the chances that exactly 5 of them will be winning numbers?
  - \*(e) Bonus question: find online the rules for another NY lottery, “Mega Millions”, and analyze your chances to win.
6. In poker, players are drawing “hands” (combinations of 5 cards) from the 52-card deck (4 suits, 13 cards in each).
  - (a) How many possible hands are there?
  - (b) How many hands in which all cards are spades?
  - (c) What are your chances of drawing a hand in which all cards are spades?
  - (d) What are your chances of drawing a hand which has 4 queens in it? [Hint: how many such hands are there?]
  - (e) What are your chances of drawing a hand which has exactly 3 queens in it?
  - (f) What are your chances of drawing a royal flush (Ace, King, Queen, Jack, 10 — all of the same suit)? [Hint: what are your chances of drawing a royal flush in a given suit, say spades?]
7. We toss a coin 100 times.
  - (a) What is the probability of obtaining all tails? exactly 2 heads? exactly 50 heads? at least 1 head?
  - (b) Same question for an unfair coin, which gives heads with probability  $p = 0.45$  and tails with probability  $q = 0.55$ .
8. A *monomial* is a product of powers of variables, i.e. an expression like  $x^3y^7$ .
  - (a) How many monomials in variables  $x, y$  of total degree of exactly 15 are there? (Note: this includes monomials which only use one of the letters, e.g.  $x^{15}$ .)
  - (b) Same question about monomials in variables  $x, y, z$ . [Hint: if you write 15 letters in a row, you need to indicate where  $x$ ’s end and  $y$ ’s begin — you can insert some kind of marker to indicate where it happens.]
  - (c) How many monomials in variables  $x, y$  of degree at most 15 are there?
  - \*(d) How many monomials in variables  $x, y, z$  of degree at most 15 are there?