

MATH 8: EUCLIDEAN GEOMETRY OVERVIEW

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1. POINTS AND LINES

Axiom 1 (Lines). For any two distinct points A, B , there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).

Axiom 2 (Distance). If points A, B, C are on the same line, and B is between A and C , then $AC = AB + BC$.

Axiom 3 (Angles). If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to 180° .

Axiom 4 (Euclid's Parallel Postulate). Let line l intersect lines m, n and angles $\angle 1, \angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called *alternate interior angles*). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.

Theorem 1. If lines l, m intersect, then they intersect at exactly one point.

Theorem 2 (Parallelism). If $l \parallel m$ and $m \parallel n$, then $l \parallel n$.

Theorem 3 (Vertical Angles). Let M be the intersection point of line segments $\overline{AB}$ and $\overline{CD}$. Then $m\angle AMC = m\angle BMD$ (such a pair of angles are called *vertical*).

Theorem 4 (Perpendicularity). Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to 90° . Then the three other angles are also equal to 90° . (In this case, we say that lines l, m are *perpendicular* and write $l \perp m$.)

Theorem 5. Let l_1, l_2 be perpendicular to m . Then $l_1 \parallel l_2$. Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Theorem 6. Given a line l and point P not on l , there exists a unique line m through P which is parallel to l .

Theorem 7. Given a line l and a point P not on l , there exists a unique line m through P which is perpendicular to l .

2. TRIANGLES AND CONGRUENCE

Definition 1 (Triangles). A triangle consists of three points (called the vertices of the triangle) and their three corresponding line segments (the sides of the triangle). A triangle with vertices A, B, C is written $\triangle ABC$, and has sides $\overline{AB}, \overline{BC}$, and $\overline{CA}$, and interior angles $\angle ABC, \angle BCA, \angle CAB$.

Definition 2 (Altitude, Median, Angle Bisector). In triangle $\triangle ABC$,

- The altitude from A is perpendicular to BC (it exists and is unique by Theorem 7).
- The median from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC).
- The angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC , then $m\angle BAE = m\angle EAC$).

Definition 3 (Isosceles Triangles). A triangle is *isosceles* if two of its sides have equal length. The two sides of equal length are called *legs*; the point where the two legs meet is called the *apex* of the triangle; the other two angles are called the *base angles* of the triangle; and the third side is called the *base*.

Definition 4 (Congruence). Congruence between two objects of the same type indicates the following:

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D , then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong \overline{CD}$.
- If two triangles $\triangle ABC, \triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}, \overline{CA} \cong \overline{FD}, \angle ABC \cong \angle DEF, \angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Definition 5 (Locus). A locus is the set of all points satisfying a certain property.

The concept of a locus is quite versatile; we will use it to describe lines and circles, but other figures can be described as well. For example, given two points A, B , the locus of points P whose sum of distances to A, B is a fixed constant (i.e. $AP + BP = c$ for some c) is called an *ellipse*. But that is a fun fact, and we won't dive into this type of figure in our geometry journey for now.

Axiom 5 (SAS Congruence). *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Axiom 6 (ASA Congruence). *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

Axiom 7 (SSS Congruence). *If two triangles have three sides congruent, then the triangles are congruent.*

Theorem 8 (Triangle Angle Sum). *The sum of the three interior angles of any triangle is 180° .*

Theorem 9 (Isosceles Base Angles). *If $\triangle ABC$ is isosceles, with base AC , then $m\angle A = m\angle C$. Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC .*

Theorem 10 (Isosceles Median, Altitude, Angle Bisector). *If B is the apex of the isosceles triangle ABC , and BM is the median, then BM is also the altitude, and is also the angle bisector, from B .*

Theorem 11 (Greater Sides Opposite Greater Angles). *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Theorem 12 (Greater Angles Opposite Greater Sides). *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

Theorem 13 (The triangle inequality). *In $\triangle ABC$, we have $AB + BC > AC$.*

Theorem 14 (Perpendicular Bisector). *The locus of points equidistant from a pair of points A, B is a line l which is perpendicular to $\overline{AB}$ and goes through the midpoint of AB . This line is called the **perpendicular bisector** of $\overline{AB}$.*

Theorem 15 (Angle Bisector). *For an angle ABC , the locus of points inside the angle which are equidistant from the two sides BA, BC is the ray $\overrightarrow{BD}$ which is the angle bisector of $\angle ABC$.*

Theorem 16. *In any triangle, the following results hold.*

1. *The perpendicular bisectors of three sides meet at a single point.*
2. *The three angle bisectors meet at a single point.*
3. *The three altitudes meet at a single point.*

3. QUADRILATERALS

Definition 6 (Quadrilaterals). A figure with four vertices, four corresponding sides, and four enclosed angles is called a **quadrilateral**; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex A is opposite vertex C). Additionally, a quadrilateral is called

- a **parallelogram**, if both pairs of opposite sides are parallel.
- a **rhombus**, if all four sides have the same length.
- a **trapezoid**, if one pair of opposite sides are parallel. (these sides are called **bases**) and the other pair is not.
- a **rectangle**, if all its angles are equal (and thus equal to 90°).

Theorem 17 (Parallelogram). *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*

Theorem 18 (Parallelogram 2).

1. Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = DC$, $AD = BC$. Then $ABCD$ is a parallelogram.
2. Let $ABCD$ be a quadrilateral such $AB = DC$, and $\overline{AB} \parallel \overline{DC}$. Then $ABCD$ is a parallelogram.

Theorem 19 (Rhombus). *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*

4. CIRCLES

Definition 7.

- A circle ω with center at O and radius r is the set of all points P such that $OP = r$.
- A radius is any line segment from O to a point A on ω
- A chord is any line segment between distinct points A, B on ω
- A diameter is a chord that passes through O ,
- A tangent line is a line that intersects the circle exactly once; if the intersection point is A , the tangent is said to be the tangent through A .

Theorem 20 (Tangent line). *Let A be a point on circle ω centered at O , and m a line through A . Then m is tangent to ω if and only if $m \perp \overline{OA}$. Moreover, there is exactly one tangent to ω at A .*

Theorem 21 (Chord perpendicular bisector). *Let $\overline{AB}$ be a chord of circle ω with center O . Then O lies on the perpendicular bisector of $\overline{AB}$. Moreover, if C is on $\overline{AB}$, then C bisects $\overline{AB}$ if and only if $\overline{OC} \perp \overline{AB}$.*

Theorem 22. *Let ω_1, ω_2 be circles with centers at points O_1, O_2 that intersect at points A, B . Then $\overline{AB} \perp \overline{O_1O_2}$.*

Theorem 23 (Tangent circles). *Let ω_1, ω_2 be circles that are both tangent to line m at point A . Then these two circles have only one common point, A . Such circles are called tangent.*

Theorem 24 (Inscribed angle). *Let A, B, C be on circle λ with center O . Then $\angle ACB = \frac{1}{2}\angle AOB$. The angle $\angle ACB$ is said to be inscribed in λ .*

5. CONSTRUCTION

Definition 8. Euclidean is a straightedge-compass construction game app, available on Google Play and the App Store; you can read about it at <https://www.euclidean.xyz> and https://twitter.com/euclidean_app

Definition 9. A straightedge is a tool that can construct a line through any two points; a compass is a tool that can construct a circle centered at a given point with radius congruent to a given line segment. A construction is a process whereby one constructs a geometric object by using tools to create other objects and then constructing any desired intersection points of the objects; a straightedge-compass construction is a construction using only straightedge and compass as tools.

In general, to provide a straightedge-compass construction of a figure, you may use any previous straightedge-compass constructions you have made as tools - for example, you may construct a parallel to any line through any point, since we have proven that a straightedge-compass construction of this is possible for any line and point.

So far, we have constructed the following items with straightedge and compass, which you may use in your constructions:

- A line segment $\overline{CD}$ congruent to $\overline{AB}$ given the points A, B , and C ;
- An angle at a given ray congruent to another given angle;
- A line parallel to a line l and through a point P ;
- A line perpendicular to a line l and through a point P ;
- The perpendicular bisector of a line segment $\overline{AB}$;
- The center of a circle;
- The medians, altitudes, and angle bisectors of a triangle;
- A circle that passes through three given points.

6. HOMEWORK - GEOMETRY

1. Prove that a diameter is the longest chord of a circle (i.e., any chord that is not a diameter has smaller length than a diameter).
2. Must a quadrilateral with three congruent sides be a rhombus? Must a quadrilateral with two right angles be a rectangle?
3. Given a triangle $\triangle ABC$,
 - (a) Provide a straightedge-compass construction of a triangle $\triangle ABC'$ congruent to $\triangle ABC$;
 - (b) Provide a straightedge-compass construction of a triangle $A'B'C'$ congruent to ABC such that corresponding sides of the two triangles are parallel;
 - (c) Prove that $A'C$ bisects AC' and vice-versa.
4.
 - (a) Prove that a line cannot intersect a circle at three distinct points.
 - (b) Prove that two circles cannot intersect at three distinct points.

7. HOMEWORK - LOGIC

1. Suppose you know that addition and multiplication satisfy the distributive property, namely that for any numbers x, y, z , we have $x(y + z) = xy + xz$. Also, suppose you know that multiplication and addition also have two other basic properties: they are commutative and associative (commutative means that $x + y = y + x$ and $xy = yx$, associative means that $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$). Now, suppose I give you two special numbers named α and β such that the following three statements are true: $\forall x(\alpha + x = x)$, $\forall x(\beta x = x)$, $\forall x(\exists y(x + y = \alpha))$. Prove the following (remember that when you see statements connected by logical connectors, it may help to understand the individual component statements first before you understand the entire thing):
 - (a) $\alpha + \alpha = \alpha$
 - (b) $\forall x((x + x = x) \implies (x = \alpha))$
 - (c) $\beta\beta = \beta$ (here I use $\beta\beta$ to indicate multiplication, though this more commonly written as β^2)
 - (d) $\forall x(((xx = x) \wedge (\exists y(xy = \beta))) \implies (x = \beta))$
 - *(e) $\forall x(\alpha x = \alpha)$
 - *(f) If $\alpha = \beta$, then all numbers are equal to α .
2. On the island of Knights and Knaves, you are greeted by your tour guides Anisha and Beatrice. Anisha tells you that they are a knight, but Beatrice tells you that Anisha is a knave. You are fed up by now with the island of knights and knaves, and simply want to make it to your hotel to check in and then enjoy the beach. There are two paths into the island, and you know that they lead in different directions and your hotel is at the end of one of them - fortunately, you are clever enough to figure out a way to get on the right path while avoiding talking to Anisha and Beatrice any more than necessary. Come up with a single question that you can ask either Anisha or Beatrice (you get to pick who) that will inform you which path is the one to your hotel, and thus allow you to ditch your guides.
3. Suppose you have a function (defined on the real numbers) that satisfies the following property for any pair of numbers x, y : $f(xf(y)) = yf(x) + 1 - y$. Is it possible to determine any values of f for certain? Try plugging in some numbers and/or variables and see what happens.
- *4. Four people sit in a room, each of them wearing a nametag given to them. The nametag is supposed to have their name on it, but some of them are blank. Nobody is allowed to look at their own nametag, so the people with blank tags don't know that theirs is blank. Nobody talks to each other (or communicates at all), other than being allowed to read everyone else's name tags. A moderator enters the room and tells the four people that if any of them figure out that their name tag is blank, they are allowed to leave the room within one minute. It turns out, all their nametags are blank. The moderator says: at least one of your tags is blank. Assuming everyone is logical and wants to leave the room (but those with actual name tags may not leave), can they figure out that their tags are all blank?