MATH 8: EUCLIDEAN GEOMETRY 4

FEB 2, 2020

Special quadrilaterals

In general, a figure with four sides (and four enclosed angles) is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex A is opposite vertex C). In case it is unclear, we use 'opposite' to refer to pieces of the quadrilateral that are on opposite sides, so side $\overline{AB}$ is opposite side $\overline{CD}$, vertex A is opposite vertex C, angle $\angle A$ is opposite angle $\angle C$ etc.

Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

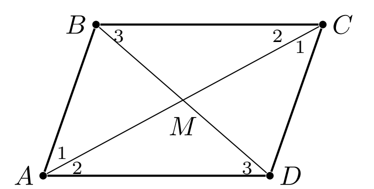
Definition 1. A quadrilateral is called

- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length
- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

Theorem 17. *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*



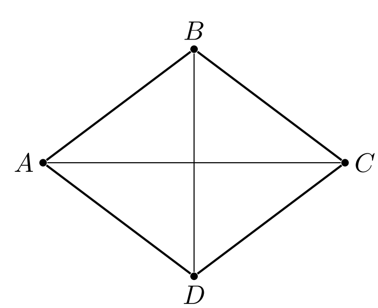
Proof. Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By Axiom 4 (alternate interior angles), angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ABC \cong \triangle CDA$. Therefore, $AB = DC$, $AD = BC$, and $m\angle B = m\angle D$. Similarly one proves that $m\angle A = m\angle C$.

Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by Axiom 4, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $BM = MD$. □

Theorem 18. *Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = DC$, $AD = BC$. Then $ABCD$ is a parallelogram.*

Proof is left to you as a homework exercise.

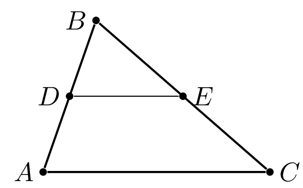
Theorem 19. *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*



Proof. Since the opposite sides of a rhombus are equal, it follows from Theorem 18 that the rhombus is a parallelogram, and thus the diagonals bisect each other. Let M be the intersection point of the diagonals; since triangle $\triangle ABC$ is isosceles, and BM is a median, by Theorem 13 in Assignment Euclidean Geometry 3, it is also the altitude. □

Midline of a triangle and trapezoid

Definition 2. A midline of a triangle $\triangle ABC$ is the segment connecting midpoints of two side.



Theorem 20. *If DE is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{AD} \parallel \overline{AC}$.*

The proof of this theorem is also given as a homework; it is not very easy.

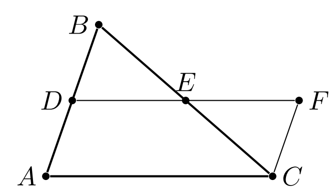
Homework

1. (Rectangle) Let $ABCD$ be a rectangle (i.e., all angles have measure 90°).
   (a) Show that opposite sides of the rectangle are congruent.
   (b) Prove that the diagonals are congruent.
   (c) Prove that conversely, if $ABCD$ is instead a parallelogram such that $AC = BD$, then it is a rectangle.

2. (Parallelogram) Who doesn't love parallelograms?
   (a) Prove Theorem 18
   (b) Prove that if in a quadrialteral $ABCD$ we have $AD = BC$, and $\overline{AD} \parallel \overline{BC}$, then $ABCD$ is a parallelogram.

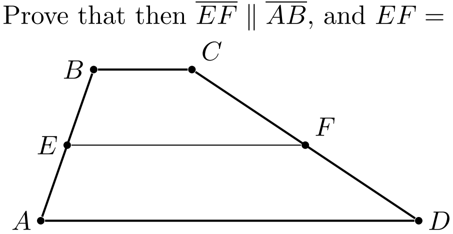
3. (Triangle Midline) Prove Theorem 20 by completing the steps below.

Continue line DE and mark on it point F such that $DE = EF$.

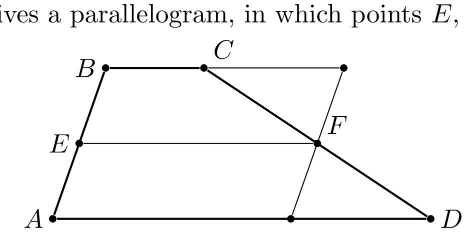


   (a) Prove that $\triangle DEB \cong \triangle FEC$
   (b) Prove that $ADFC$ is a parallelogram (hint: use alternate interior angles!)
   (c) Prove that $DE = \frac{1}{2}AC$

4. (Trapezoid Midline) Let $ABCD$ be a trapezoid, with bases AD and BC, and let E, F be midpoints of sides AB, CD respectively.
Prove that then $\overline{EF} \parallel \overline{AB}$, and $EF = (AD + BC)/2$.



[Hint: draw through point F a line parallel to AB, as shown in the figure below. Prove that this gives a parallelogram, in which points E, F are midpoints of opposite sides.]



5. (Constructions) Given a triangle $\triangle ABC$, complete (with proof) the following straightedge-compass constructions:
   (a) Construct the median from A to $\overline{BC}$
   (b) Construct the altitude from A to $\overline{BC}$
   (c) Construct the angle bisector from A to $\overline{BC}$
   (d) Now, for a slightly different exercise, given a circle, complete (with proof) a straightedge-compass construction of the center point of the circle.