MATH 8: EUCLIDEAN GEOMETRY 3

JANUARY 26, 2019

1. Triangle inequalities

In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal (Theorem 9). The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

Theorem 10. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Proof. Assume not. Then either $BC = AB$ or $BC < AB$.

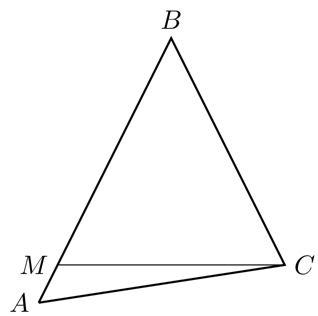
But if $BC = AB$, then $\triangle ABC$ is isosceles, so by Theorem 9, $m\angle A = m\angle C$ as base angles, which gives a contradiction.

Now assume $BC < AB$, find the point M on AB so that $BM = BC$, and draw the line MC. Then $\triangle MBC$ is isosceles, with apex at B. Hence $m\angle BMC = m\angle MCB$. On the other hand, by Problem 5, we have $m\angle BMC > m\angle A$, and by Axiom 3, we have $m\angle C = m\angle ACM + m\angle MCB > m\angle MCB$, so

$$m\angle C > m\angle MCB = m\angle BMC > m\angle A$$

so we have reached a contradiction.

Thus, assumptions $BC = AB$ or $BC < AB$ both lead to a contradiction. Therefore, the only possibility is that $BC > AB$.



□

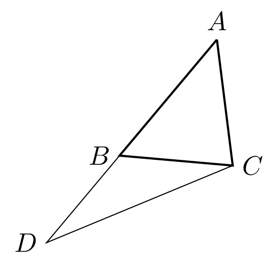
The converse of the previous theorem is also true: opposite a longer side, there must be a larger angle.The proof is left as an exercise.

Theorem 11. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 12 (The triangle inequality)**.** *In $\triangle ABC$, we have $AB + BC > AC$.*

Proof. Extend the line AB past B to the point D so that $BD = BC$, and join the points C and D with a line so as to form the triangle ADC. Observe that $\triangle BCD$ is isosceles, with apex at B; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 10, this implies $AD > AC$. Our result now follows from $AD = AB + BD$ (Axiom 2) □



2. Median, Altitude, Angle Bisector

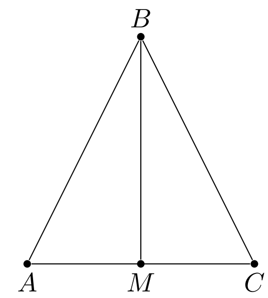
There are three special lines that can be constructed from any vertex in any triangle; each line goes from a vertex of the triangle to the line containing the triangle's opposite side (altitudes may sometimes land on the opposite side outside of the triangle).

Given a triangle $\triangle ABC$,

- The altitude from A is the line through A perpendicular to $\overleftrightarrow{BC}$;
- The median from A is the line from A to the midpoint D of $\overline{BC}$;
- The angle bisector from A is the line $\overleftrightarrow{AE}$ such that $\angle BAE \cong \angle CAE$. Here we let E denote the intersection of the angle bisector with $\overline{BC}$.

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide.

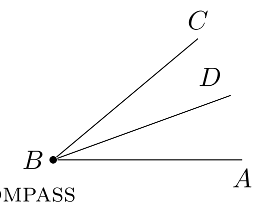
Theorem 13. *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*



Proof. Consider triangles $\triangle ABM$ and $\triangle CBM$. Then $AB = CB$ (by definition of isosceles triangle), $AM = CM$ (by definition of midpoint), and $m\angle MAB = m\angle MCB$ (by Theorem 9). Thus, by SAS axiom, $\triangle ABM \cong \triangle CBM$. Therefore, $m\angle ABM = m\angle CBM$, so BM is the angle bisector.
Also, $m\angle AMB = m\angle CMB$. On the other hand, $m\angle AMB + m\angle CMB = m\angle AMC = 180^\circ$. Thus, $m\angle AMB = m\angle CMB = 180^\circ/2 = 90^\circ$. □

The following result is an analog of Theorem 13. For a point P and a line l, we define the distance from P to l to be the length of the perpendicualr dropped from P to l (see problem 1 in the HW). We say that point P is equidistant from two lines l, m if the distance from P to l is equal to the distance from P to m.

Theorem 14. *For an angle ABC, the locus of points inside the angle which are equidistant from the two sides BA, BC is the ray $\overrightarrow{BD}$ which is the angle bisector of $\angle ABC$.*



Proof of this theorem is given as a homework.

3. Constructions with straightedge and compass

Now that we know when two geometric objects are the same (via congruence), it makes sense to ask if we can produce figures with specific properties of interest — for example, if we can reproduce a given angle somewhere else so that the resulting angle is congruent to the original. Traditionally, such constructions are done using straight-edge and compass: the straight-edge tool constructs lines and the compass tool constructs circles. More precisely, it means that we allow the following basic operations:

- Draw (construct) a line through two given or previously constructed distinct points. (Recall that by axiom 1, such a line is unique).
- Draw (construct) a circle with center at previously constructed point O and with radius equal to distance between two previously constructed points B, C
- Construct the intersections point(s) of two previsly constructed lines, circles, or a circle and a line

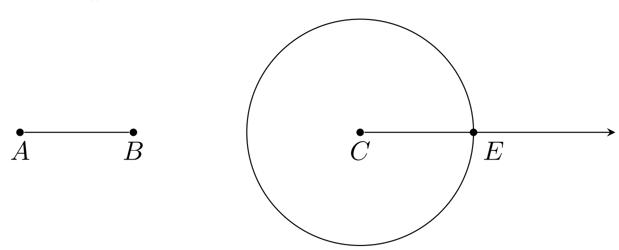
All other constructions (e.g., draw a line parallel to a given one) must be done using these elementary constructions only!!

Constructions of this form have been famous since mathematics in ancient Greece.

Here are some examples of constructions:

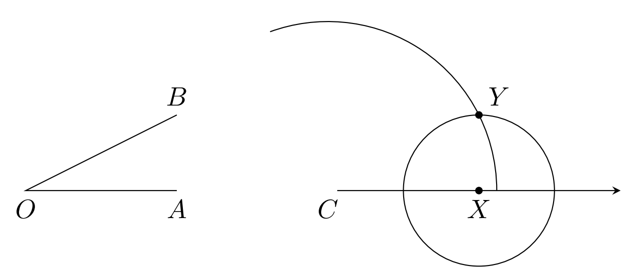
Example 1. Given any line segment $\overline{AB}$ and ray $\overrightarrow{CD}$, one can construct a point E on $\overrightarrow{CD}$ such that $\overline{CE} \cong \overline{AB}$.

Construction. Construct a circle centered at C with radius AB. Then this circle will intersect $\overrightarrow{CD}$ at the desired point E. □



Example 2. Given angle $\angle AOB$ and ray $\overrightarrow{CD}$, one can construct an angle around $\overrightarrow{CD}$ that is congruent to $\angle AOB$.

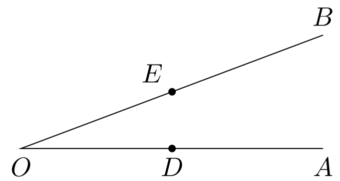
Construction. First construct point X on $\overrightarrow{CD}$ such that $CX \cong OA$. Then, construct a circle of radius OB centered at C and a circle of radius AB centered at X. Let Y be the intersection of these circles; then $\triangle XCY \cong \triangle AOB$ by SSS and hence $\angle XCY \cong \angle AOB$. □



4. Homework

1. (Midline!) There is another special line of a triangle, though it's less common and generally less useful. But, here is a problem about it anyways.

(a) Given line segments $\overline{OA}$ and $\overline{OB}$ and midpoint D of $\overline{OA}$, prove that a point E on $\overline{OB}$ is the midpoint of $\overline{OB}$ if and only if $\overline{DE} \parallel \overline{AB}$.



(b) Given triangle $\triangle ABC$, let D, E be the midpoints of sides $\overline{AB}$, $\overline{AC}$ respectively. Prove that $\overline{DE} \parallel \overline{BC}$ and $DE = \frac{1}{2}BC$. (The line segment $\overline{DE}$ is called the midline of the triangle from A.)

(c) Given a triangle $\triangle ABC$ and point D on $\overline{AC}$, prove that the midlines of $\triangle ABD$ and $\triangle CBD$ are parallel.

2. (Congruence Juggling) There will be a lot of triangles in this problem, many more than you are given. You're going to have to make them yourself. Good luck!

(a) Let $\triangle ABC$ be a right triangle with right angle $\angle A$, and let D be the midpoint of $\overline{BC}$. Prove that $AD = \frac{1}{2}BC$.

(b) Prove Theorem 14. [If you happen to know hypotenuse-leg congruence, try to find a proof without using it.]

3. (Circumcenter & Incenter) In this problem you will prove that a triangle's perpendicular bisectors all intersect at a point, and a triangle's angle bisectors all intersect at a point. These points have special names, but I will not tell you why (until next week)! Have fun!

(a) Let l_1, l_2 be the perpendicular bisectors of side AB and BC respectively of $\triangle ABC$, and let F be the intersection point of l_1 and l_2. Prove that then F also lies on the perpendicular bisector of the side BC. [Hint: use Theorem 13.]

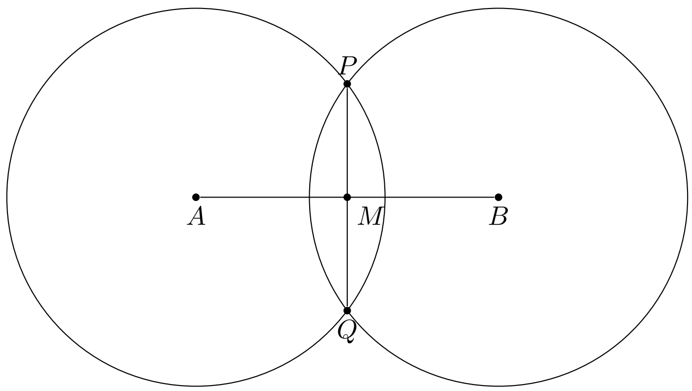
(b) Let the angle bisectors from B and C in the triangle $\triangle ABC$ intersect each other at point F. Prove that $\overleftrightarrow{AF}$ is the third angle bisector of $\triangle ABC$. [Hint: use Theorem 14]

4. (Point to Line) Suppose you want to know how far a point is from a line. How would you figure this out? In this problem you will prove both that the perpendicular from a point to a line is the shortest distance from that point to the line, and that you can construct such a perpendicular with straightedge and compass.

(a) Let P be a point not on line l, and $A \in l$ be the base of perpendicular from P to l: $AP \perp l$. Prove that for any other point B on l, $PB > PA$ ("perpendicular is the shortest distance"). Note: you can not use Pythagorean theorem as we have not proved it yet; instead, try using Theorem 11.

(b) The following method explains how one can find the midpoint of a segment AB using a ruler and compass:

- Choose radius r (it should be large enough) and draw circles of radius r with centers at A and B.
- Denote the intersection points of these circles by P and Q. Draw a line $\overleftrightarrow{PQ}$.
- Let M be the intersection point of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$. Then M is the midpoint of AB.



Can you justify this method, i.e., prove that so constructed point will indeed be the midpoint of AB? You can use the defining property of the circle: for a circle of radius r, the distance from any point on this circle to the center is exactly r. [Hint: find some isosceles triangles!]

(c) Given a point P and a line l, construct the perpendicular to l through P. [Hint: a circle centered at P may help.]

5. (Orthocenter) In this problem, you will work with the altitudes of a triangle. It is more of a guided proof than the other problems, which basically means it looks longer but is less work. Of course, if you don't read the problems carefully, then you wouldn't know that, but if you're reading this right now then you do. Anyways, here you go.

Let $\triangle ABC$ be isosceles with $\overline{AB} \cong \overline{AC}$. Let the altitudes from B and C intersect their opposite legs at the points D and E respectively, and let $\overline{BD}$, $\overline{CE}$ intersect at F.

(a) Prove $\angle EBF \cong \angle DCF$

(b) Prove $\triangle DBC \cong \triangle ECB$

(c) Prove $\triangle DCF \cong \triangle EBF$

(d) Prove $\triangle AEF \cong \triangle ADF$

(e) Prove that $\overleftrightarrow{AF}$ is the third altitude of $\triangle ABC$.