MATH 8: ASSIGNMENT 1

JANUARY 12, 2020

Euclidean geometry tries to describe geometric properties of various figures in the plane. Figures are understood as sets of points; we will use capital letters for points and write $P \in m$ for "point P lies in figure m", or "figure m contains point P". The notion of "point" can not be defined: it is so basic that it is impossible to explain it in terms of simpler notions. In addition, there are some other basic notions (lines, distances, angles) that can not be defined. Instead, we can state some basic properties of these objects; these basic properties are usually called "postulates" or "axioms of Euclidean geometry". **All results in Euclidean geometry should be proved by deducing them from the axioms**; justifications "it is obvious", "it is well-known", or "it is clear from the figure" are not acceptable.

We allow use of all logical rules. We will also use all the usual properties of real numbers, equations, inequalities, etc.

For your enjoyment, take a look at the book which gave rise to Euclidean geometry and much more, Euclid's *Elements*, dated about 300 BC, and used as the standard textbook for the next 2000 years. Nowadays it is avaialable online at `http://math.clarku.edu/~djoyce/java/elements/toc.html`

1. Basic objects

These objects are the basis of all our constructions: all objects we will be discussing will be defined in terms of these objects. No definition is given for these basic objects.

- Points
- Lines
- Distances: for any two points A, B, there is a non-negative number AB, called distance between A, B.
- Angle measures: for any angle $\angle ABC$, there is a real number $m\angle ABC$, called the measure of this angle (more on this later).

We will assume that every line has infinitely many points on it. Also, we will assume that any nontrivial angle has positive measure.

We will also frequently use words "between" when describing relative position of points on a line (as in: A is between B and C) and "inside" (as in: point C is inside angle $\angle AOB$). We do not give full list of axioms for these notions; it is possible, but rather boring.

Having these basic notions, we can now define more objects. Namely, we can give definitions of

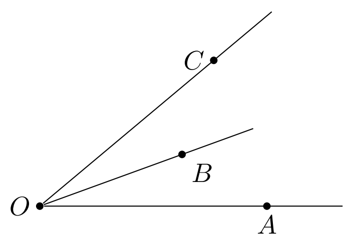
- interval, or line segment (notation: $\overline{AB}$): set of all points on line AB which are between A and B, together with points A and B themselves
- ray (notation: $\overrightarrow{AB}$): set of all points on the line AB which are on the same side of A as B
- angle (notation: $\angle AOD$): figure consisting of two rays with a common vertex
- parallel lines: two distinct lines l, m are called parallel (notation: $l \parallel m$) if they do not intersect, i.e. have no common points. We also say that every line is parallel to itself.

2. First postulates

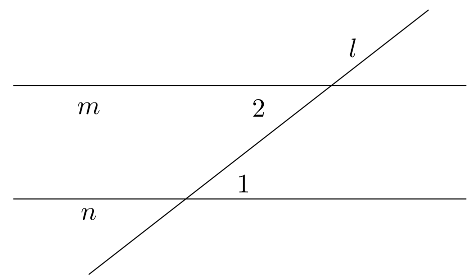
Axiom 1. *For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

Axiom 2. *If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*



Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called* alternate interior angles*). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



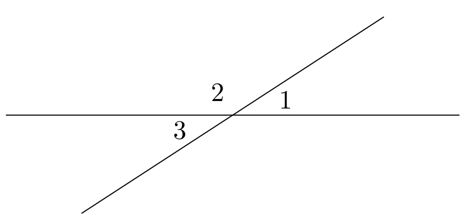
3. First theorems

Theorem 1. *If lines l, m intersect, than they intersect at exactly one point.*

Proof. Assume that they intersect at more than one point. Let P, Q be two of the points where they intersect. Then both l, m go through P, Q. This contradicts Axiom 1. Thus, our assumption (that l, m intersect at more then one point) must be false. □

Theorem 2. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Theorem 3. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in the figure below (such a pair of angles are called* vertical*). Then $m\angle 1 = m\angle 3$.*



Proof. Let angle 2 be as shown in the figure to the left. Then, by Axiom 3, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. □

Theorem 4. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are* perpendicular *and write $l \perp m$.)*

Theorem 5. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Theorem 6. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

4. Triangles

Theorem 8. *Given any three points A, B, C, which are not on the same line, and line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, we have $m\angle ABC + m\angle BCA + m\angle CAB = 180°$. (Such a figure of three points and their respective line segments is called a* triangle*, written $\triangle ABC$. The three respective angles are called the triangle's* interior angles.*)*

5. Congruence

It will be helpful, in general, to have a way of comparing geometric objects to tell whether they are the same. We will build up such a notion and call it congruence of objects. To begin, we define congruence of angles and congruence of line segments (note that an angle cannot be congruent to a line segment; the objects have to be the same type).

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D, then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong \overline{CD}$.
- If two triangles $\triangle ABC$, $\triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Note that congruence of triangles is sensitive to which vertices on one triangle correspond to which vertices on the other. Thus, $\triangle ABC \cong \triangle DEF \implies \overline{AB} \cong \overline{DE}$, and it can happen that $\triangle ABC \cong \triangle DEF$ but $\neg(\triangle ABC \cong \triangle EFD)$.

6. Congruence of Triangles

Triangles consist of six pieces (three line segments and three angles), but some notion of constancy of shape in triangles is important in our geometry. We describe below some rules that allow us to, in essence, uniquely determine the shape of a triangle by looking at a specific subset of its pieces.

Axiom 5 (SAS Congruence)**.** *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Other congruence rules about triangles follow from the above: the ASA and SSS rules. However, their proofs are less interesting than other problems about triangles, so we can take them as axioms and continue.

Axiom 6 (ASA Congruence)**.** *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

Axiom 7 (SSS Congruence)**.** *If two triangles have three sides congruent, then the triangles are congruent.*

7. Isosceles triangles

A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

Theorem 9 (Base angles equal)**.** *If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.*

Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

Proof. Assume that $\triangle ABC$ is isoceles, with apex B. Then by SAS, we have $\triangle ABC \cong \triangle CBA$. Therefore, $m\angle A = m\angle C$.

The proof of the converse statement is left to you as a homework exercise. □

8. Homework

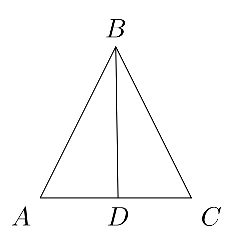
Note that you may use all results that are presented in the previous sections. This means that you may use Theorem 3, for example, if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

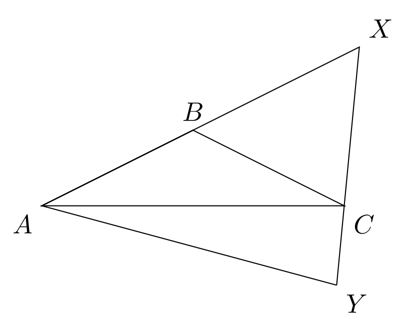
1. (Parallel and Perpendicular Lines) Part of the spirit of Euclidean geometry is that parallelism and perpendicularity are special concepts; Theorem 6, for example, is generally considered part of the heart of Euclidean geometry. For this problem, prove the following theorems presented in the First Theorems section, using only the information from the Basic Objects and First Postulates sections. Axiom 4 will be of key importance.

(a) Prove Theorem 2. [Hint: assume that l and n are not parallel; then they must intersect at some point P.]
(b) Prove Theorem 4.
(c) Prove Theorem 5.
(d) Prove Theorem 6. (You may assume that at least one such parallel line exists, you must prove that there can't be more than one.)
(e) Prove Theorem 7. (You may assume that at least one such perpendicular line exists, you must prove that there can't be more than one.)

2. (Isosceles Triangles) Isosceles triangles have particularly useful symmetry properties; the fact that congruent sides comes hand in hand with congruent angles is essential to understanding isosceles triangles. For this problem, you are given isosceles triangle $\triangle ABC$, with base $\overline{AC}$. The first part of the problem will ask you to look inside the triangle, and the second part will create some structures outside the triangle; figures are given below to help you, if you want them.

(a) Suppose D is a point on $\overline{AC}$ such that $\overline{AD} \cong \overline{DC}$. Prove that $\angle DBA \cong \angle DBC$. What can we say about $\angle ADB$?
(b) Extend $\overline{AB}$ to point X, and then extend $\overline{XC}$ to point Y. Prove that $\angle XAC \cong \angle CAY$ if and only if $\overline{BC} \parallel \overline{AY}$.





3. (Triangle Congruence) Recall that when we refer to triangles, in particular triangle congruence, it is significant the order in which we label the vertices. In general (architecture, drawing, origami etc.), one thinks of triangles as meaningful in their own right without vertex ordering, but in geometry, points themselves are quite significant, so we do need to point them out. For this problem, you are given triangle $\triangle ABC$.

(a) Explain why $\triangle BCA$ and $\triangle BAC$ are also triangles.

(b) Prove that, if $\triangle ABC \cong \triangle CBA$, then $\triangle ABC$ is isosceles.

(c) Prove the converse of Theorem 9, i.e. if $\triangle ABC$ has $\angle A \cong \angle C$, then $\triangle ABC$ is isosceles.

(d) If $\triangle ABC \cong \triangle BCA$, what can we say about the side lengths of this triangle?

(e) Suppose D is a point on side $\overline{AB}$. Is it possible for $\triangle ABC$ and $\triangle ADC$ to be congruent? What if I let you pick the order of the vertices?

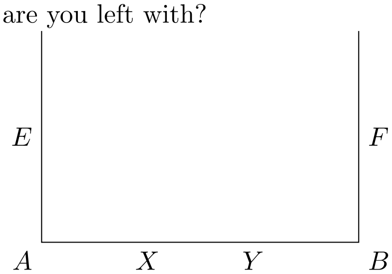
4. (Stop!) Although a partial figure is provided, this problem may be an interesting exercise in imagination. Have fun!

(a) Suppose A, B, C, D are points such that $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$ and $\angle DAB \cong \angle ABC \cong \angle BCD \cong \angle CDA$. Prove that each of the four angles is a right angle. What is this figure called?

(b) Suppose X, Y are on $\overline{AB}$ (with X closer to A and Y closer to B, as shown in the figure below), and E is on $\overline{AD}$ and F on $\overline{BC}$. Prove that $m\angle EXY > 90°$.

(c) Suppose $m\angle EXY = 135°$. Prove that $\triangle EAX$ is isosceles.

(d) Suppose now that $m\angle EXY = 135°$, $\triangle EAX \cong \triangle FBY$, and $\overline{EX} \cong \overline{XY}$. Suppose also that two triangles are constructed around vertices C and D as well, essentially forming two more corner triangles, each congruent to $\triangle EAX$. Now cut all four of these triangles out from $ABCD$. What are you left with?



5. (Elephants and Hamsters) Work on any two problems from the Elephants and Hamsters homework sheet that we have not solved in class.