MATH 8: SPECIAL

JANUARY 5, 2020

1. Hello!

Good evening, students of math. As there are not many of us remaining in class for this semester, we can try some more interactive material. As such, I decided to show you the Elephant and Hamsters theory, one which requires nontrivial, sometimes difficult, combination of abstract axioms in order to deduce results. Let's get started.

2. Axioms

- (Friendship Axiom) In our universe, there are Elephants and there are Hamsters. Elephants may be friends with Hamsters, Hamsters may be friends with Elephants, but Elephants are never friends with each other, and Hamsters are never friends with each other. Friendship is mutual, i.e. if Elephant A is friends with Hamster H, then H is friends with A.
- (Mutual Friendship Axiom) Any two distinct Hamsters have a mutual Elephant friend; any two distinct Elephants not in the same family have a mutual Hamster friend. Elephants in the same family have no mutual friends.
- (Family Drama Axiom) If Elephant A and Hamster H are not friends, then there exists an Elephant that is friends with H and family with A.
- (Theme Song Existence Axiom) Anytime $n > 1$ Elephants are all friends with Hamster H, this friendship has n theme songs. (The songs may or may not be the same.) Note that theme songs are products of the mutual friendship of a set of Elephants with a single Hamster. If A, B are the Elephants, we write $[AB]$ to refer to the set of theme songs of the mutual friendship of A,B with H, and $[AB]_1$, $[AB]_2$ to refer to the individual songs (it doesn't strictly matter what order we put the songs in, as long as we are able to identify them all).
- (Song Combination Axiom) Theme songs can be combined. Given $n > 1$ theme songs, they can be combined to produce a single, new theme song.
- (Theme Song Sharpness Axiom) Any time theme songs are combined, the new song is different from all the original songs. Additionally, if songs x and y combine to give song z, and song r is different from y, then x and r combine to something that's different from z.
- (Jingle Bells Axiom) Anytime $n > 1$ Elephants are all friends with Hamster H, the theme songs of this friendship all combine to Jingle Bells.
- (Family Gossip Axiom) If Elephants A and B are friends with Hamster H, and C is some other Elephant such that B and C are friends with Hamster J, then the theme songs of A and B with H are the same as the theme songs of B and C with J *if and only if* A is family with C.
- (Song Compression Axiom) Let Elephants A_1, ..., A_n all be friends with Hamster H. Then, for some selection of i, j from 1 to n, the theme songs $[A_iA_j]$ are the combinations of mutually exclusive nonempty subsets of $[A_1...A_n]$. Additionally, a different selection of A_i, A_j would find songs that are combinations of different subsets of $[A_1...A_n]$.

3. Example Problems

1. If H, I, and J are Hamsters, A is the mutual Elephant friend of H with I and B is the mutual Elephant friend of I with J, then prove that A and B cannot be family.
Proof: A and B are both friends with I, and by Mutual Friendship Axiom, family cannot have mutual friends.

2. If A, B, and C are Elephants such that Hamster H is mutual friends with A,B, Hamster I is mutual friends with B,C, and Hamster J is mutual friends with A,C, prove that one can select one theme song from each of the three friendships such that the three songs combine to Jingle Bells.
Proof: By Family Drama axiom, there is an Elephant D that is friends with H and family with C.
By Family Gossip, the theme songs of A,D with H are the same as those of A,C with J, i.e. $[AD] = [AC]$.
By Family Gossip, the theme songs of B,D with H are $[BC]$, i.e. $[BD] = [BC]$.
By Theme Song Existence, there are three theme songs of the friendship of A,B,D with H, in a set called $[ABD]$.
By Song Compression and Theme Song Sharpness, we can prove that one song of $[ABD]$ is the same as one of $[AB]$; similarly, a song of $[ABD]$ is one of the songs of $[BD] = [BC]$ and another song of $[ABD]$ is one of $[AD] = [AC]$. These three songs are choices of different subsets of $[ABD]$, thus must actually be different songs, thus they cover all three songs $[ABD]$. Thus one of $[AB]$, one of $[BC]$, and one of $[AC]$ form the set $[ABD]$.
By Jingle Bells, $[ABD]$ combines to Jingle Bells.
Thus, we have proved that there is some selection of one song of $[AB]$, one of $[BC]$, and one of $[AC]$ that combines to Jingle Bells.

3. Suppose A and B are Elephants, and E is an elephant that is not in the same family as A or B. Suppose also that $[AE]_1 + [BE]_1 =$Jingle Bells. Prove that A, B are in the same family.
Proof: By Jingle Bells axiom, we know that $[AE]_1 + [AE]_2 =$Jingle Bells.
We are given that $[AE]_1 + [BE]_1 =$Jingle Bells.
By the second part of Theme Song Sharpness, we deduce that $[AE]_2 = [BE]_1$.
By Jingle Bells axiom, $[BE]_1 + [BE]_2 =$Jingle Bells. By similar logic to above, we can prove that $[BE]_2 = [AE]_1$.
Thus the sets $[AE]$ and $[BE]$ are the same two theme songs, and therefore, by Family Gossip axiom, A and B must be family.

4. This is just a reminder on set theory. Suppose X, Y are sets such that $X = X \cap Y$. Prove then that $X \subset Y$.
Proof: Suppose some element r is in X, i.e. $r \in X$. Then because $X = X \cap Y$, we have that $r \in (X \cap Y)$. By the definition of intersection, we deduce that r is in both X and Y, the relevant point being $r \in Y$. Thus we have deduced that if r is in X, then r is in Y, i.e. $(r \in X) \implies (r \in Y)$. This is the definition of subset, thus we deduce $X \subset Y$. The concept at work here is that, if X loses nothing when it intersects with Y, then everything in X had to be in Y to begin with.

4. Practice Problems

When solving these problems, you are allowed to use the results of the example problems in the previous section (for example, example problem number 2 may be helpful for practice problem number 6 below).

1. If A, B, C are Elephants that are all friends with Hamster H, prove that $[ABC]_1 + [ABC]_2$ cannot equal Jingle Bells.

2. If A, B, C are Elephants that are all friends with Hamster H, prove that one of the theme songs of $[AB]$ must be the same as one of the theme songs of $[ABC]$. Prove additionally that $[AB] \cap [ABC]$ (set intersection) has exactly one song in it.

3. Suppose Elephant A is friends with Hamster H but *not* with Hamster J. Let B be the Family Drama Elephant that is friends with J and family with A. Use one of the axioms to prove that there must be an Elephant that is mutual friends of H, J; which axiom did you use? Can this new Elephant be family with A or B? Which axiom answers this question?

4. Arrange 5 Elephants and 5 Hamsters so that all Elephants are friends with exactly 2 of the Hamsters and all Hamsters are friends with exactly 2 of the Elephants. Draw a diagram if it helps.

5. If A, B, C are Elephants that are all friends with Hamster H, prove the following:
   (a) $[AB] \cap [ABC] \neq [BC] \cap [ABC]$
   (b) $[AB] \cap [ABC] \neq [AC] \cap [ABC]$
   (c) $[BC] \cap [ABC] \neq [AC] \cap [ABC]$
   (d) $[ABC] \subset ([AB] \cup [BC] \cup [AC])$

6. Let A, B, C, D be Elephants such that AB are not family, BC are not family, CD are not family and AD are not family. Suppose $[AB] = [CD]$ and $[BC] = [AD]$. Prove that AC are family.

7. Let A, B, C be Elephants not in the same family. Let H, I, J be Hamsters that are mutual friends with A, B ; B, C ; A, C respectively. Let D be some Elephant that is *not* friends with H. Given $[AC]_1 = [AD]_1$, prove that $[BC] = [BD]$.

8. Prove that if $A_1 ... A_n$ are Elephants that are all friends with Hamster H, and X is some proper subset of $[A_1 ... A_n]$ (*proper subset* means a subset that is not equal to the entire set), then the songs in X do not combine to Jingle Bells.

9. Suppose A, B, C, D are Elephants, all from different families. Prove that one can select a song from $[AB]$, one from $[BC]$, one from $[CD]$ and one from $[AD]$ such that these four songs combine to (Jingle Bells+Jingle Bells). You can call this song *Double Jingle*.

10. Is it possible to put together a collection of n Elephants and n Hamsters such that, for some integer $2 < k < n$, every Elephant in the collection is friends with exactly k hamsters, and every Hamster in the collection is friends with exactly k Elephants?