MATH 8
THE MATH BATTLE!
NOV 17, 2019

1. Jane has baked a cake and wants to cut it into pieces (not necessarily equal) so that it can be evenly divided among 8 people or among 10 people. What is the smallest possible number of pieces?

16: 8 pieces of size $\frac{1}{10}$*, and 8 pieces of size* $\frac{1}{40}$*. To verify this works, first check that* $8 \cdot \frac{1}{10} + 8 \cdot \frac{1}{40} = \frac{32}{40} + \frac{8}{40} = 1$*, then check that for ten people, eight of them take a* $\frac{1}{10}$ *slice and the last two take four of the* $\frac{1}{40}$ *slices; for eight people, each person takes one* $\frac{1}{10}$ *slice and one* $\frac{1}{40}$ *slice.*
To prove that this is minimal, if there are less than sixteen slices, then when dividing to eight people, one of these people must get only one slice, therefore there must be a slice of size $\frac{1}{8}$*. However, this slice is unusable when dividing the cake among ten people, as any one person getting this slice would get too much cake - therefore the existence of a slice of this size is a contradiction.*

2. Solve

$$x = \sqrt{20} + \cfrac{13}{\sqrt{20} + \cfrac{13}{\sqrt{20} + \cfrac{13}{\sqrt{20} + \cfrac{13}{\sqrt{20} + \cfrac{13}{x}}}}}$$

There are several ways to do this, but a simple one is to let α *be a number that satisfies* $\alpha = \sqrt{20} + \frac{13}{\alpha}$*; then simply substitute* $\alpha = \sqrt{20} + \frac{13}{\alpha}$ *into the right-hand side of* $\alpha = \sqrt{20} + \frac{13}{\alpha}$ *and repeat as desired. The solution is then achieved by solving* $\alpha = \sqrt{20} + \frac{13}{\alpha}$ *for* α*.*

3. 20 computers are connected by cables so that:

- each cable connects two computers
- each computer has at most two cables attached

A system administrator wants to label the cables using color labels so that for each computer, the cables attached to it have different colors. What is the minimal number of colors which would be enough for any configuration of computers?

Pick a computer and follow one of its cables to another computer, then that computer has at most one more cable going out of it - if it does have a cable, follow that to another computer, and repeat this process until you run into a computer with only one cable or one of the original computers (which must be the first one you started with, else you'd have a computer with three cables). Thus you get either a line segment or a loop of connected computers. Repeat this until all computers are accounted for - this process proves that the arrangement must be a union of loops and line segments. It remains to prove that any loop or line segment can be colored with three colors of cables, which is not too hard, simply alternate the colors, using the third color if you end up with an odd cycle that requires it. To prove that two colors is not enough, make a triangle of three interconnected computers, this obviously cannot be colored with two colors (the remaining 17 computers can remain unconnected, doesn't matter).

4. I have several identical pieces of candy that I wish to distribute among my three children. Assuming I am unbothered by shameless favoritism, yet somehow still hold to the moral principle that no child may receive more than five pieces of candy, prove that the number of ways to distribute five pieces of candy among my three children is equal to the number of ways to distribute ten pieces of candy among my three children.

Start each kid off with 5 pieces of candy and then distribute five pieces of 'negative candy' to each kid. This is identical to distributing ten pieces of candy to the three kids with a *maximum of five to each kid. This proves the correspondence. The key is to think of the two scenarios as complements or negatives of each other, in a sense.*

5. In a regular 5000-gon, 2001 vertices are colored in red. Show that it is possible to choose 3 red vertices which form an isosceles triangle.

The detail of 2001 being just one more than 2/5 of 5000 is the intuitive key to the problem, but here's the formal solution: in a regular pentagon, any three vertices form an isosceles triangle; in a regular 5000-gon, the polygon is effectively a collection of 1000 regular pentagons, similar to the way a regular hexagon can be reduced to a union of two equilateral triangles - you'll lose the original edges, but you'll keep all the vertices and they'll all be in exactly the same place. Then, of these 1000 pentagons, by pigeonhole principle, one of them must have more than 2 red vertices.

6. On the Island of Knights and Knaves, there happens to be a welcoming party on the dock awaiting your arrival. They are to help you get to your hotel on the island, but they decide to greet you with the following statements:

- B says: "A would say B would say A would say B would say A would say B would say A would say B would say A would say C would say A and B would say A is a knave."
- A says: "B would say C would say B would say C would say A would say C would say B would say H would say A would say H would say B would say B would say A would say H is a knight^{note1}."
- C says: "A would say H and B are knights and D is a knave."
- H says: "C would say D^{note2} is a knight."
- D says: "C and D would say H would say C would say A would say H would say I is a knave."
- I says: "D would say D would say I would say D would say B would say A would say B would say A would say B would say B would say A would say Q^{note3} is a knave."
- M says: "B is a knave, H is a knight and D would say A is a knave."
- S says: "Everyone except me is a knave."
- D2 says: "D is a knave."
- R2D2 says: "The force is with you."
- Q says: "Follow me and I will lead you to your place of residence for your stay. Also, I'm a generally trustworthy person."

[1]original problem said knave, but that leads to a contradiction.
[2]original problem had B here instead of D, but this leads to a contradiction.
[3]original problem said B here instead of Q, but this leads to redundant (yet miraculously logically consistent) information.

First of all, lol. Second, note that 'X would say something' in knights and knaves terms is the logical equivalence relation, as are the initial 'says' parts of the statements. We will use this to render the statements in logical form - before we do that, though, we will make use of the following result:
$X \iff (Y \iff Z)$ is equivalent to $X+Y+Z$, where $+$ indicates the logical XOR relation.
While $\iff$ is not associative, $+$ is, so this allows us to write without parentheses. These results will not be proved (I'll leave it to you).
We also use the minor clarification that "X says (Y and Z would say this)" means "X says Y would say this $\wedge$ X says Z would say this", and accordingly split all such statements into two component and statements and simplify.
We now simplify the points, using the logical variables X to mean "X is a knight", and for brevity $=$ to mean $\iff$. What results is a list of necessarily true statements:

- $B = (A + B + A + B + A + B + A + B + A + C + (A = \neg A \wedge B = \neg A))$
- $A + B + C + B + C + A + C + B + H + A + H + B + B + A + H$
- $C + A + (H \wedge B \wedge \neg D)$
- $H + C + D$
- $D = (C = (H + C + A + H + \neg I)) \wedge D = (D = (H + C + A + H + \neg I))$
- $I + D + D + I + D + B + A + B + A + B + B + A + \neg Q$
- $M = (\neg B \wedge H \wedge (D = \neg A))$
- S says: "Everyone except me is a knave."
- $D2 = \neg D$
- $R2D2 =$The force is with you
- $Q =$Q is your guide

Again we simplify, now using the facts that $(X + X \iff False)$, $(X + False \iff X)$, $(X = \neg Y \iff (X + Y))$, $(\neg X \iff True + X)$, *and* $(X = Y \iff True + X + Y)$*:*

- $B = (A + C + False)$ which simplifies to $True + B + A + C$
- $C + B + H$
- $C + A + (H \wedge B \wedge \neg D)$
- $H + C + D$
- $(D + C + C + A + I + True) \wedge (D + D + C + A + I + True)$ which simplifies to $(D + A + I + True) \wedge (C + A + I + True)$
- $D + A + Q + True$
- $M = (\neg B \wedge H \wedge (D + A))$
- S says: "Everyone except me is a knave."
- $D2 = \neg D$
- $R2D2 =$The force is with you
- $Q =$Q is your guide

We will now make logical deductions - recall that a logical deduction of a statement X *is a result of* X *being true, but this deduction does not have to be equivalent to* X*!*
From $X \wedge Y$ *we can deduce* $X = Y$*, as indeed both* X *and* Y *must be true. Thus from* $(D + A + I + True) \wedge (C + A + I + True)$ *we deduce* $D + A + I + True = C + A + I + True$ *which simplifies to* $D = C$.
Next we use the fact that $C + A + I + True$ *is true to deduce that* $C + A + I$ *is false, and the sum of a false statement and a true statement is a true one, therefore* $(True + B + A + C) + (C + A + I)$ *must be true, from which we deduce* $True + B + I$*, which is equivalent to* $B = I$.
We have deduced $C = D$ *and* $B = I$*. Also, "The force is with you"=True. We can now substitute these in all our statements and simplify.*

- $True + B + A + C$
- $C + B + H$
- $C + A + (H \wedge B \wedge \neg C)$
- $H + C + C$ which simplifies to H
- $(C + A + B + True) \wedge (C + A + B + True)$ which simplifies to $True + A + B + C$, which is now redundant
- $True + A + C + Q$
- $M = (\neg B \wedge H \wedge (C + A))$
- S says: "Everyone except me is a knave."
- $D2 = \neg C$
- $R2D2 = True$
- $Q =$Q is your guide

We can now substitute $H = True$ in our statements. Also, since S says everyone except S is a knave, and we have deduced that $H = True$ i.e. H is a knight, we deduce that S must be a knave.

- $True + B + A + C$; from below $True + C + B$ we deduce that $C + B$ is false and hence adding it to this statement should not affect veracity, thus we get $True + B + A + C + C + B$ is true, which simplifies to $True + A$, i.e. A is a knave
- $True + C + B$
- $C + A + (H \wedge B \wedge \neg C)$, substituting $H = True$ and $A = False$ we get $C + (B \wedge \neg C)$ which miraculously (I'll leave it up to you) simplifies to $B \vee C$, also congratulations to symbol "$\vee$" on your first appearance in the entire problem
- H
- $True + A + C + Q$, substituting $A = False$ we get $True + C + Q$
- $M = (\neg B \wedge C)$; notice that $(\neg B \wedge C) \wedge (B \vee C) \iff (\neg B)$, so this simplifies to $M = \neg B$ or equivalently $M + B$
- $\neg S$
- $D2 = \neg C$
- $R2D2$
- Q =Q is your guide

We can now list the true statements we have: $\neg A$, $True + C + B$, $B \vee C$, H, $True + C + Q$, $M + B$, $\neg S$, $D2 = \neg C$, $R2D2$, and Q's statement.
I point you now to the logical deduction $(True + C + B) \wedge (B \vee C)$ deduces $(B \wedge C)$, which thence deduces both B and C. From here we get $D2 = \neg C = False$ and $M + B = M + True$, from which we deduce $\neg D2$ and $\neg M$.
Lastly, coming to a clean(ish) finish, we use $(True + C + Q) \wedge C$ to deduce Q, thus resulting in our final lineup:

- A is a knave.
- B is a knight.
- C is a knight.
- D is lost to the wind.
- H is a knight.
- I is forgotten to the sands of time.
- S is a knave.
- R2D2 is a knight.
- Q is actually telling the truth lol