MATH 8
ASSIGNMENT 5: LOGIC PT. 1 - TREES
OCTOBER 21, 2018

In a strange universe far, far away, there are mysterious objects, which, through careful observation, we see are connected somehow by links. The links, however, seem to abide by some code of patterns! Some testing is conducted, and we have determined the following rules. For convenience, we will refer to our objects with letters such as a and b, and we will denote the links with the symbol $<$.

- *Rule 1: the linking is asymmetric, and has a direction. If $a < b$, then $b \not< a$.*

- *Rule 2: the linking appears to be transitive - namely, if we have three objects a, b, c, and $a < b$ and $b < c$, then $a < c$.*

- *Rule 3: no object is linked to itself. That is, for all possible a, $a \not< a$.*

Equipped with this knowledge, we wish to investigate further into this universe. Let us travel there now, and attempt to discover what rules we can figure out about these objects, and what patterns they can make!

1. In how many distinct ways can three objects be connected by links?

2. Suppose I have three objects a, b, c.
  (a) If $b < c$ and $a < b$, according to the rules, how many objects must a be linked to?
  (b) Is it possible for a to be linked to all three objects a, b, c? [Hint: Use the third rule]

3. Can you think of a way to draw objects and links that follows the rules? Draw a few collections of four objects connected by links of your choosing.

4. Suppose I have four objects a, b, c, d.
  (a) If $a < b$ and $b < d$ and $a < c$ and $c < d$, must a be linked to d?
  (b) If now instead we have $b < c$ and $b < d$ and $a < b$, then which objects is a linked to?

5. Is it possible for three items to link in a loop? Specifically, if we have objects a, b, c, is it possible that we have the three links $a < b$, $b < c$, $c < a$? What about four objects - can we have $a < b < c < d < a$?

6. You stumble upon a strange object which appears to be the bottom of something - there is nothing linked to it from below. You find that this object links to one object above it, which then links to one object above that, and so on, and each new object you find links to one object above.
  (a) Draw a diagram of what this collection of objects might look like.
  (b) You decide you want to name all the objects in this collection you have just discovered. Starting from the base object and continuing to all the others, can you think of a naming system that might be suitable?

7. I have found a small, interesting area of this universe in which every object is connected to every other object (so, for example, we have no split branches).
  (a) Conceptualize and/or draw a few diagrams of object arrangements in which every object is linked to every other. Do you notice any patterns?
  (b) If I have four objects in which every object is linked to every other, how many different configurations can these objects take? (Are there multiple possible linking patterns that satisfy this rule? Be sure to include examples and proofs.)

***8.** We have just engineered a scanning device that can tell us all the objects that are linked to a given object from below. We will call this the object's *scan*. If an object's scan turns up nothing (i.e. if no objects are linked to it from below), we'll say the object has no scan.
If, given an object a, all of the objects in the scan of a are also all linked to each other, then we say that a, together with its scan, are a *tree branch*. If a tree branch contains an object with no scan, that object is said to be the *root* of the branch. A branch with a root is a *rooted branch*.
A collection of objects in which every object is on a rooted tree branch is called a *tree*.

(a) If there is some object in a collection that is in the scan of every other object in that collection, must the collection be a tree?

(b) If I take two different trees and put them side by side and declare this to be a new collection (no new links are formed), is this collection of two trees itself a tree?

(c) Create a clear definition of what it means, in your understanding, for a tree to be connected.

(d) Devise a rule to determine when a tree is connected according to your definition.

9. I decide to play a game with a pile of 20 stones, just for fun. In this game, I can split the pile into two smaller piles, and record the result on my blackboard as follows: I begin by putting the number '0' on my blackboard, and when I split the stone pile, I multiply the sizes of the resulting piles and add that number to my blackboard's number. After doing the first split, I then choose one of the remaining piles, and perform the same operation - split it into smaller piles and multiply the product of the sizes of the resulting piles and add that into the number on my blackboard. Here's an example: I can start by splitting (20) into $(5, 15)$, giving me 75 on the blackboard; then I split the 15 to get $(5, 3, 12)$, giving me $75 + 36 = 111$ on my blackboard; then split the 5 to get $(1, 4, 3, 12)$, giving me $111 + 4 = 115$ on my blackboard; etc.
I continue this game until I'm left with 20 piles of size 1. Is there a strategy I can adopt to maximize the final number that I get on my blackboard? [Hint: try playing with smaller starting pile sizes and see if you notice a pattern.]