MATH 8
ASSIGNMENT 4: BINOMIAL THEOREM

OCT 6, 2019

Announcements

We will be participating in the AMC8 olympiad on Nov 12. If you are interested, please fill out the form linked in an announcement on the SchoolNova website homepage.

Main formulas of combinatorics

Recall the numbers ${}_nC_k$ from Pascal's triangle:

$$\begin{aligned} {}_nC_k &= \text{ The number of paths on a chessboard going } k \text{ units up and } n-k \text{ units to the right} \\ &= \text{The number of words that can be written using } k \text{ zeros and } n-k \text{ ones} \\ &= \text{The number of ways to choose } k \text{ items out of } n \textbf{ if the order does not matter} \end{aligned}$$

We have discussed the following formula for them:

$$ {}_nC_k = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!k!} \tag{1}$$

Binomial formula

These numbers have one more important application:

$$(a+b)^n = {}_nC_0 a^n + {}_nC_1 a^{n-1}b^1 + \cdots + {}_nC_n b^n \tag{2}$$

The general term in this formula looks like ${}_nC_k \cdot a^{n-k}b^k$. For example, for $n = 3$ we get

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

(compare with the 3rd row of Pascal's triangle)

This formula is called the **binomial formula**; we discussed its proof today.

Problems

In all the problems, you can write your answer as a combination of factorials, ${}_nC_k$, and other arithmetic – you do not have to do the computations. As usual, please write your reasoning, not just the answers!

1. Use the binomial formula to expand the following expressions:
  (a) $(x-y)^3$
  (b) $(a+3b)^3$
  (c) $(2x+y)^5$
  (d) $(x+2y)^5$

2. Find the coefficient of x^8 in the expansion of $(2x+3)^{14}$

3. Compute $(1+\sqrt{3})^6 + (1-\sqrt{3})^6$

4. Compute $(x+2y)^6 - (x-2y)^6$

5. Show that $(1+\sqrt{3})^{12} + (1-\sqrt{3})^{12}$ is integer.

6. Deduce that Pascal's triangle is symmetric, i.e. ${}_nC_k = {}_nC_{n-k}$ in two ways:

(a) Using the binomial formula for $(x+y)^n$ and $(y+x)^n$.
(b) Using formula (1).

7. (a) Use the binomial formula to compute
  (i) Sum of all numbers in the nth row of Pascal's triangle. [Hint: take $a = b = 1$ in the binomial formula.]
  (ii) Alternating sum of all numbers in the nth row of Pascal's triangle: ${}_nC_0 - {}_nC_1 + {}_nC_2 - {}_nC_3 \dots$
(b) Recall in Homework 2, Problem 8 when we first looked at summing the rows of Pascal's triangle. Determine a way of proving the above result about alternating sums without using the binomial formula.

8. Let p be prime.
(a) Show that each of the binomial coefficients ${}_pC_k$, $1 \le k \le p-1$, is divisible by p.
(b) Show that if a, b are integer, then $(a+b)^p - a^p - b^p$ is divisible by p.

***9.** Long ago, the four nations decided to hold a relay race competition. Forty-eight people signed up, twelve from each of four element-nations: Water, Earth, Fire, Air; however a relay run consists of four people, so only sixteen of those people can compete.
(a) Given that each nation must select four people to form a team, how many ways can this be done?
(b) Now consider they run the competition slightly differently: teams will consist of one person from each nation, and four teams will be chosen. How many ways can this be done?

10. [Some of you may have seen this — but not all of you...]
(a) Given a group of 25 people, we ask each of them to choose a day of the year (non-leap, so there are 365 possible days). How many possible combinations can we get? [Order matters: it is important who has chosen which date]
(b) The same question, but now we additionally require that all chosen dates be different.
(c) In a group of 25 people, what are the chances that no two of them have their birthday on the same day? Conversely, what is the chance that at least two people have the same birthday?

1. Miscellaneous

1. Answer the following without using a calculator or computer.
(a) (AMC) Order the following numbers from smallest to largest: 10^8, 5^{12}, 2^{24}.
(b) Order the following numbers from smallest to largest: $(3+\sqrt{3})^4$, $(2+\sqrt{3})^5$, $3^6 + 3^5$

2. A group of elves is famous for their ability to run in perfect synchronization with each other - in fact, they always run at exactly the same speed anytime they run. There is a local track in town, and out of the group of elves, any number of them may show up to the track on a particular day, during which they will space themselves evenly throughout the track and run clockwise at their standard speed.
On Sunday, seven elves show up to the track, as does Roshni; Roshni runs one lap counterclockwise. As Roshni is running opposite the direction of the elves, Roshni passes by an elf more than seven times: in fact, Roshni passes by an elf exactly nine times.
On Monday, four elves show up to the track, as does Ana; Ana runs one lap counterclockwise, passing elves eight times.
On Tuesday, eleven elves show up to the track, as does Jun; Jun runs one lap counterclockwise, passing elves sixteen times.
Rank Roshni, Ana, and Jun from fastest to slowest average running speed for the laps they ran.

3. You are given a regular tetrahedron of side length 5, a regular octahedron of side length 1, and a regular cube of side length 1. A blue sphere is to be placed inside the tetrahedron so as to maximize its volume; similarly a pink sphere of maximum volume inside the octahedron, and a purple sphere of maximum volume inside the cube. Rank the blue, pink, and purple spheres from least to greatest volume.