MATH 8
ASSIGNMENT 3: FORMULA FOR ${}_nC_k$
SEP 29, 2019

Main formulas of combinatorics

Recall the numbers ${}_nC_k$ from the Pascal triangle:

$$
\begin{aligned}
{}_nC_k &= \text{ The number of paths on the chessboard going } k \text{ units down and } n-k \text{ to the right} \\
&= \text{The number of words that can be written using } k \text{ zeros and } n-k \text{ ones} \\
&= \text{The numberof ways to choose } k \text{ items out of } n \text{ if the } \textbf{order does not matter}
\end{aligned}
$$

It turns out that there is an explicit formula for them:

$$ {}_nC_k = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!k!} $$

Thus, we now have a full list of all the main formulas of combinatorics:

- The number of ways to order k items is
$$ k! = k(k-1)\cdots 2\cdot 1 $$
- The number of ways to choose k items out of n if the **order matters** is
$$ {}_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!} $$
- The number of ways to choose k items out of n if the **order does not matter** is
$$ {}_nC_k = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{(n-k)!k!} $$

${}_nC_k$ Problems

In all the problems, you can write your answer as a combination of factorials and ${}_nC_k$ – you do not have to do the computations. And, as usual, please write your reasoning, not just the answers!

1. A senior class in a high school, consisting of 120 students, wants to choose a class president, vice-president, and 3 steering committee members. How many ways are there for them to do this?

2. How many five letter "words" are there such that the letters are in alphabetical order? (Here a "word" is any sequence of letters from the alphabet a through z.)

3. Andrew has 7 pieces of candy, and Tim has 9 (all different). They want to trade 5 pieces of candy. How many wasy are there for them to do it?

4. If you have 5 lines on the plane so that no two are parallel and there are no triple intersection points, how many triangles do they form? What if there are n lines?

5. In one of the lotteries run by New York State, "Sweet Million", they randomly choose 6 numbers out of numbers 1–40. If you guess all 6 correctly (order does not matter), you win $1,000,000. [There are also smaller prizes for guessing 5 out of 6, etc., but let us ignore them for now.]
(a) How many ways are there to choose 6 numbers out of 40?
(b) What are your chances of winning?
(c) If a lottery ticket cost $1, how much money does New York State make for each ticket sold (on average)?
*(d) If you choose 6 numbers out of 40 at random, what are the chances that exactly 5 of them will be winning numbers?

Bonus question: find online the rules for another NY lottery, “Mega Millions”, and analyze your chances to win.

6. In poker, players are drawing “hands” (combinations of 5 cards) from the 52-card deck (4 suits, 13 cards in each).
   (a) How many possible hands are there?
   (b) How many hands in which all cards are spades?
   (c) What are your chances of drawing a hand in which all cards are spades?
   (d) What are your chances of drawing a hand which has 4 queens in it? [Hint: how many such hands are there?]
   (e) What are your chances of drawing a hand which has exactly 3 queens in it?
   (f) What are your chances of drawing a royal flush (Ace, King, Queen, Jack, 10 — all of the same suit)? [Hint: what are your chances of drawing a royal flush in a given suit, say spades?]

7. We toss a coin 100 times.
   (a) What is the probability of obtaining all tails? exactly 2 heads? exactly 50 heads? at least 1 head?
   (b) Same question for an unfair coin, which gives heads with probability $p = 0.45$ and tails with probability $q = 0.55$.

8. A *monomial* is a product of powers of variables, i.e. an expression like x^3y^7.
   (a) How many monomials in variables x, y of total degree of exactly 15 are there? (Note: this includes monomials which only use one of the letters, e.g. x^{15}.)
   (b) Same question about monomials in variables x, y, z.[Hint: if you write 15 letters in a row, you need to indicate where x's end and y's begin — you can insert some kind of marker to indicate where it happens.]
   (c) How many monomials in variables x, y of degree at most 15 are there?
   *(d) How many monomials in variables x, y, z of degree at most 15 are there?

1. More Combinatorics

1. A frog on vacation to India is attempting to climb a step at the Taj Mahal. The frog, figuring it has a one in ten chance to succeed at the jump, decides to attempt the jump ten times (if it succeeds early, it won't make further attempts). What is the chance that the frog will make it up the step after its ten attempts?

2. Three rooks are said to be *friendly* if they are in three distinct but consecutive rows on the chessboard. How many ways are there to put three rooks on a chessboard so that they are friendly? (Assume the rooks are all the same color.)

3. Let L be a square lattice of side length 1 - i.e., the set of all points (a, b) on the xy coordinate plane such that a and b are integers. Let M be a similar square lattice but of side length 3, and possibly with a different center - i.e., M is the set of all points $(n + c, m + d)$ where c, d are integer multiples of 3 and n, m are a given pair of integers. Notice that, for example, if $n = m = 3$, the resulting lattice M is the same as when $n = m = 0$ - we say in this case that these two placements of M on L are identical. How many distinct placements of M are there on L?

2. Miscellaneous

1. A function f on the integers (or in general) is said to be *involutive* if $f(f(n)) = n$ for all n. Can you find an involutive function that satisfies $f(0) = 1$?

2. Is it possible to make a single slice in a tetrahedral block of cheese so that one of the faces of the remaining block (without the slice) is a rectangle?

3. Plot the following x, y equations on an xy coordinate plane:
   (a) $x^2 + y^2 - 1 = 0$
   (b) $x^2 + (y - 1)^2 - 1 = 0$
   (c) $(x^2 + y^2 - 1)(x^2 + (y - 1)^2 - 1) = 0$
   (d) $(x^2 + y^2 - 1)^2 + (x^2 + (y - 1)^2 - 1)^2 = 0$