## RECIPROCAL FUNCTION FAMILY

Functions that model inverse variation have the form $f(x) = \frac{a}{x}$, where $x \neq 0$. They belong to a family whose parent is the **reciprocal function** $f(x) = \frac{1}{x}$, where $x \neq 0$.

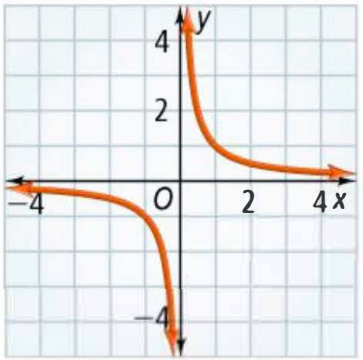
### TAKE NOTE Key Concept

### General Form of the Reciprocal Function Family

The general form of a member of the reciprocal function family is $y = \frac{a}{x - h} + k$, where $x \neq h$.

The inverse variation functions, $y = \frac{a}{x}$, are stretches, shrinks (or compressions), and reflections of the parent reciprocal function, depending on the value of $a$.

The graph of the parent reciprocal function $y = \frac{1}{x}$ is shown at the right.



## Problem 1 Graphing an Inverse Variation Function

**What is the graph of $y = \frac{8}{x}, x \neq 0$? Identify the *x*- and *y*-intercepts and the asymptotes of the graph. Also, state the domain and range of the function.**

**THINK**

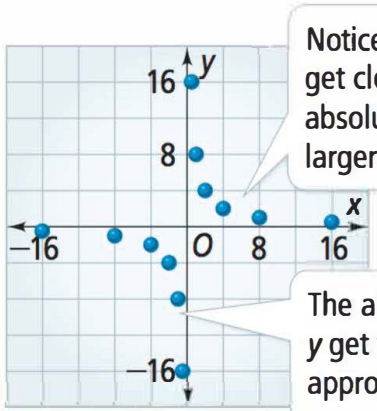
**What values should you choose for *x*?**

Choose values of *x* that divide nicely into 8. Make a table of points that are easy to graph.

**Step 1** Make a table of values that includes positive and negative values of *x*.

| *x* | *y* | *x* | *y* |
|---|---|---|---|
| −16 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 16 |
| −8 | −1 | 1 | 8 |
| −4 | −2 | 2 | 4 |
| −2 | −4 | 4 | 2 |
| −1 | −8 | 8 | 1 |
| $-\frac{1}{2}$ | −16 | 16 | $\frac{1}{2}$ |

**Step 2** Graph the points.




Notice how the *y*-values get closer to zero as the absolute values of *x* get larger.

The absolute values of *y* get very large as *x* approaches zero.

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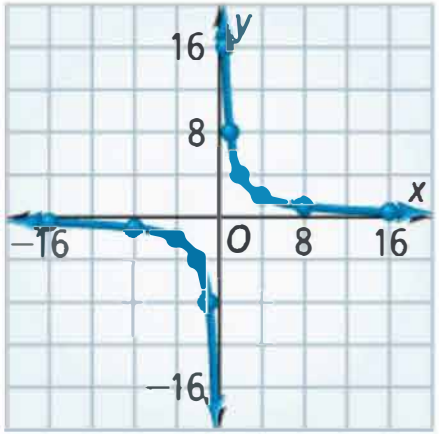
**Step 3** Connect the points with a smooth curve. $x$ cannot be zero, so there is no $y$-intercept.

The numerator is never zero, so $y$ is never zero. There is no $x$-intercept.

The $x$-axis is a horizontal asymptote. The $y$-axis is a vertical asymptote. Knowing the asymptotes provides you with the basic shape of the graph.

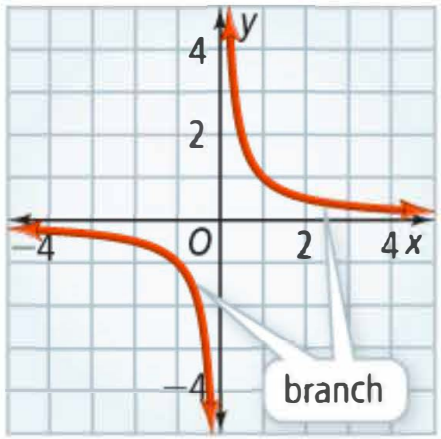


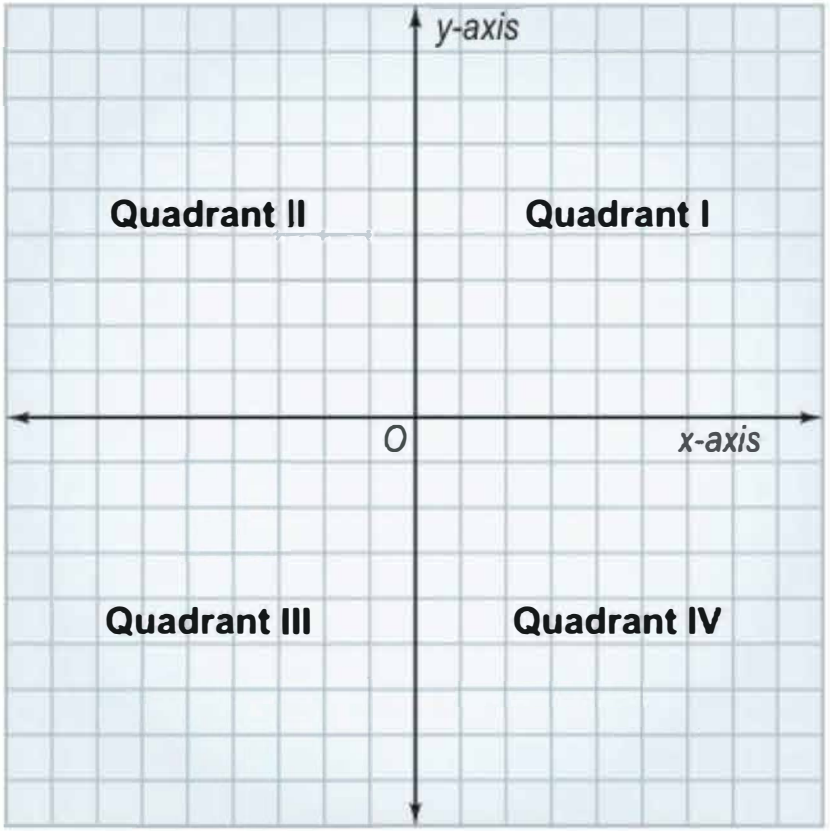

The domain is the set of all real numbers except $x = 0$. The range is the set of all real numbers except $y = 0$.

GOT IT?

## REMEMBER THIS

Each part of the graph of a reciprocal function is a **branch**. The branches of the parent function $y = \frac{1}{x}$ are in Quadrants I and III. Stretches and compressions of the parent function remain in the same quadrants. Reflections are in Quadrants II and IV. The graph of a reciprocal function has two parts



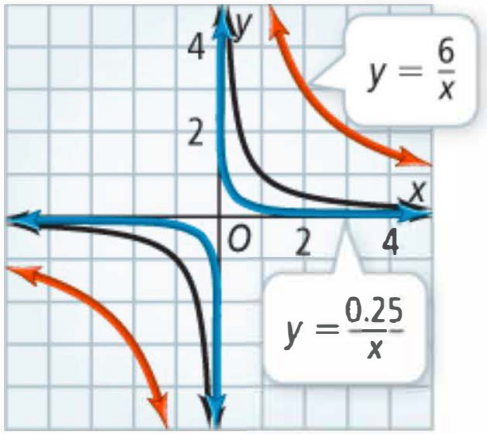

y-axis
Quadrant II
Quadrant I
O
x-axis
Quadrant III
Quadrant IV

## Problem 2 Identifying Reciprocal Function Transformations

**For each given value of *a*, how do the graphs of $y = \frac{1}{x}$ and $y = \frac{a}{x}$ compare? What is the effect of *a* on the graph?**

**A** $a = 6$

The graph (in red) of $y = \frac{6}{x}$ is a stretch of the graph of $y = \frac{1}{x}$ (in black) by the factor 6.

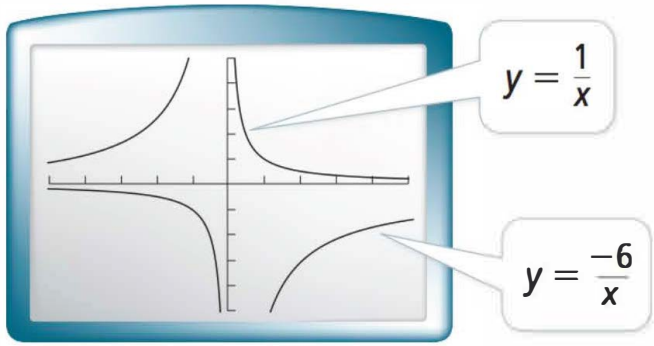



**B** $a = 0.25$

The graph (in blue) of $y = \frac{0.25}{x}$ is a shrink of the graph of $y = \frac{1}{x}$ (in black) by the factor $\frac{1}{4}$.

**C** $a = -6$

The graph of $y = \frac{-6}{x}$ is the stretch by the factor 6 in part A followed by a reflection across the *x*-axis.




### THINK

**How does the negative sign in part C affect the graph?**
The *y*-values have signs that are opposite those in part A. The graph in part A reflects across the *x*-axis.

SHOW SOLUTION

GOT IT?

You can translate any reciprocal function horizontally or vertically just as you can other functions.

## TAKE NOTE Key Concept

### The Reciprocal Function Family

| | |
|---|---|
| **Parent Function** | $y = \frac{1}{x}, x \neq 0$ |
| Stretch $(\lvert a \rvert > 1)$<br>Shrink $(0 < \lvert a \rvert < 1)$<br>Reflection $(a < 0)$ across $x$-axis | $y = \frac{a}{x}, x \neq 0$ |
| Translation (horizontal by $h$; vertical by $k$) with vertical asymptote $x = h$ horizontal asymptote $y = k$ | $y = \frac{1}{x - h} + k, x \neq h$ |
| Combined | $y = \frac{a}{x - h} + k, x \neq h$ |

When you graph a translated reciprocal function, a good first step is to draw the asymptotes.

## Problem 3 Graphing a Translation

**What is the graph of $y = \frac{1}{x+1} - 2$? Identify the domain and range.**

**THINK**

**How do you find the asymptotes?**

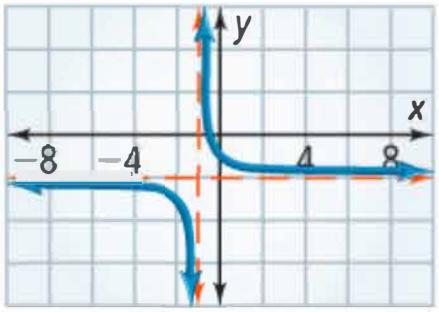
The asymptotes of $y = \frac{1}{x}$ (the axes) translate 1 unit to the left and 2 units down.

**Step 1** Draw the asymptotes (red).

For $y = \frac{1}{x+1} - 2$, $h = -1$ and $k = -2$.

The vertical asymptote is $x = -1$.

The horizontal asymptote is $y = -2$.




**Step 2** Translate the graph of $y = \frac{1}{x}$.

The graph of $y = \frac{1}{x}$ contains the points $(1, 1)$ and $(-1, -1)$. Translate these points 1 unit to the left and 2 units down to $(0, -1)$ and $(-2, -3)$, respectively. Draw the branches through these points (blue).

The domain is the set of all real numbers except $x = -1$. The range is the set of all real numbers except $y = -2$.

GOT IT?

If you know the asymptotes of the graph of a reciprocal function and the value of $a$, you can write the equation of the function.

## Problem 4 Writing the Equation of a Transformation

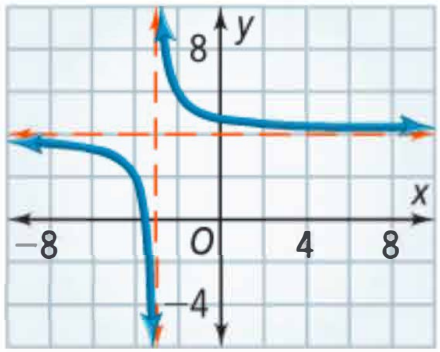
**Multiple Choice** **This graph of a function is a translation of the graph of $y = \frac{2}{x}$. What is an equation for the function?**

**A.** $y = \frac{2}{x+3} + 4$

**C.** $y = \frac{2}{x-3} + 4$

**B.** $y = \frac{2}{x+3} - 4$

**D.** $y = \frac{2}{x-3} - 4$




### PLAN

**How can you get started?**
Identify the asymptotes of the graph.

SHOW SOLUTION

GOT IT?

The asymptotes are $x = -3$ and $y = 4$. Thus $h = -3$ and $k = 4$.

$y = \frac{a}{x-h} + k$ Use the general form.

$y = \frac{2}{x-(-3)} + 4$ Substitute for $a$, $h$, and $k$.

$y = \frac{2}{x+3} + 4$ Simplify.

The correct choice is A.

## Problem 5 Using a Reciprocal Function

**Clubs** **The rowing club is renting a 57-passenger bus for a day trip. The cost of the bus is $750. Five passengers will be chaperones. If the students who attend share the bus cost equally, what function models the cost per student *C* with respect to the number of students *n* who attend? What is the domain of the function? How many students must ride the bus to make the cost per student no more than $20?**

### KNOW

- The bus holds 57 passengers.
- The bus costs $750.
- Five riders are chaperones who pay nothing for the bus.

### NEED

- A function for the cost per student
- The number of students needed so that the cost does not exceed $20 per student

### PLAN

- Write a reciprocal function for the situation.
- Graph the function and solve an inequality using the $20 limit.

To share the cost equally, divide 750 by the number of students, *n*, who attend.

The function that models the cost per student is $C = \frac{750}{n}$.

The bus has a capacity of 57 passengers and there will be 5 chaperones. The maximum number of students is $57 - 5 = 52$.

The domain is the integers from 1 to 52.

### THINK

**Is the domain $x \leq 52$?**

No; the domain is the possible numbers of students, so only positive integers make sense.

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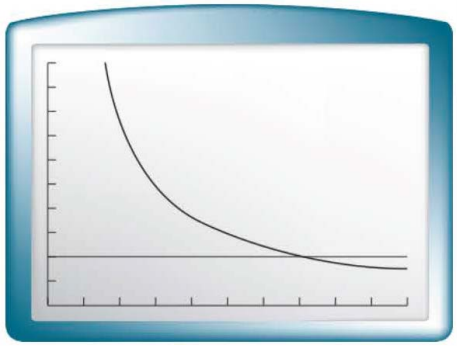
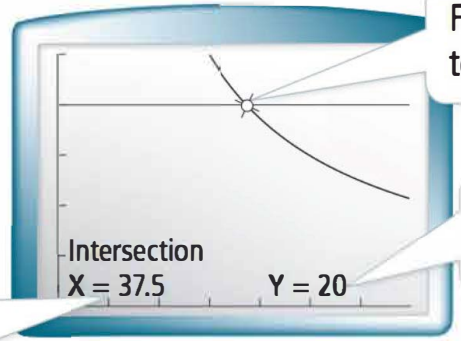
Use a graphing calculator to solve the inequality $\frac{750}{n} \leq 20$. Let $\mathbf{Y1} = \frac{\mathbf{750}}{\boldsymbol{x}}$ and $\mathbf{Y2} = \mathbf{20}$.



Change the window dimensions to get a closer look at the graph. Use the **intersect** feature.




For all values greater than or equal to 38, the cost is less than $20.

If $x = 37$, the cost will be more than $20.

The number of people must be a whole number.

At least 38 students must ride the bus.

GOT IT?

# Lesson Check

## Do you know HOW?

1. Graph the equation $y = \frac{3}{x}$.

**Describe the transformation from the graph of $y = \frac{1}{x}$ to the graph of the given function.**

2. $y = \frac{1}{x} + 5$

3. $y = \frac{-4}{x}$

4. What are the asymptotes of the graph of $y = \frac{5}{x+2} - 7$?

## Do you UNDERSTAND?

5. **Vocabulary** What transformation changes the graph of $y = \frac{1}{x}$ into the graph of $y = \frac{1}{2x}$?

6. **Open Ended** Write an equation of a stretch and a reflection of the graph $y = \frac{1}{x}$ across the $x$-axis.

7. **Writing** Explain how you can tell if a function $y = \frac{a}{x}$ is a stretch or compression of the parent function $y = \frac{1}{x}$.

# Practice and Problem-Solving Exercises

## A • Practice

**Graph each function. Identify the *x*- and *y*-intercepts and the asymptotes of the graph. Also, state the domain and the range of the function.** SEE PROBLEM 1.

8. $y = \frac{2}{x}$

9. $y = \frac{15}{x}$

10. $y = \frac{-3}{x}$

11. $y = -\frac{10}{x}$

12. $y = \frac{10}{x}$

**Graphing Calculator** **Graph the equations $y = \frac{1}{x}$ and $y = \frac{a}{x}$ using the given value of *a*. Then identify the effect of *a* on the graph.** SEE PROBLEM 2.

13. $a = 2$

14. $a = -4$

15. $a = 0.5$

16. $a = 12$

17. $a = 0.75$

## Practice and Problem-Solving Exercises - *Continued*

**Sketch the asymptotes and the graph of each function. Identify the domain and range.**
SEE PROBLEM 3.

**18.** $y = \frac{1}{x} - 3$

**19.** $y = \frac{-2}{x} - 3$

**20.** $y = \frac{1}{x-2} + 5$

**21.** $y = \frac{1}{x-3} + 4$

**22** $y = \frac{2}{x+6} - 1$

**23.** $y = \frac{10}{x+1} - 8$

**24.** $y = \frac{1}{x} - 2$

**25.** $y = \frac{-8}{x+5} - 6$

**Write an equation for the translation of $y = \frac{2}{x}$ that has the given asymptotes.** SEE PROBLEM 4.

**26.** $x = 0$ and $y = 4$

**27.** $x = -2$ and $y = 3$

**28.** $x = 4$ and $y = -8$

**29. Construction** The weight $P$ in pounds that a beam can safely carry is inversely proportional to the distance $D$ in feet between the supports of the beam. For a certain type of wooden beam, $P = \frac{9200}{D}$. What distance between supports is needed to carry 1200 lb? SEE PROBLEM 5.

## Practice and Problem-Solving Exercises - *Continued*

### B • Apply

**30.** **Think About a Plan** A high school decided to spend $750 on student academic achievement awards. At least 5 awards will be given, they should be equal in value, and each award should not be less than $50. Write and sketch a function that models the relationship between the number *a* of awards and the cost *c* of each award. What are the domain and range of the function?

- Which equation describes the relationship between *a* and *c*?
- What information can you use to determine the domain and range?

**31.** **Open-Ended** Write an equation for a horizontal translation of $y = \frac{2}{x}$. Then write an equation for a vertical translation of $y = \frac{2}{x}$. Identify the horizontal and vertical asymptotes of the graph of each function.

**Sketch the graph of each function.**

**32.** $xy = 3$

**33.** $xy + 5 = 0$

**34.** $3xy = 1$

**35.** $5xy = 2$

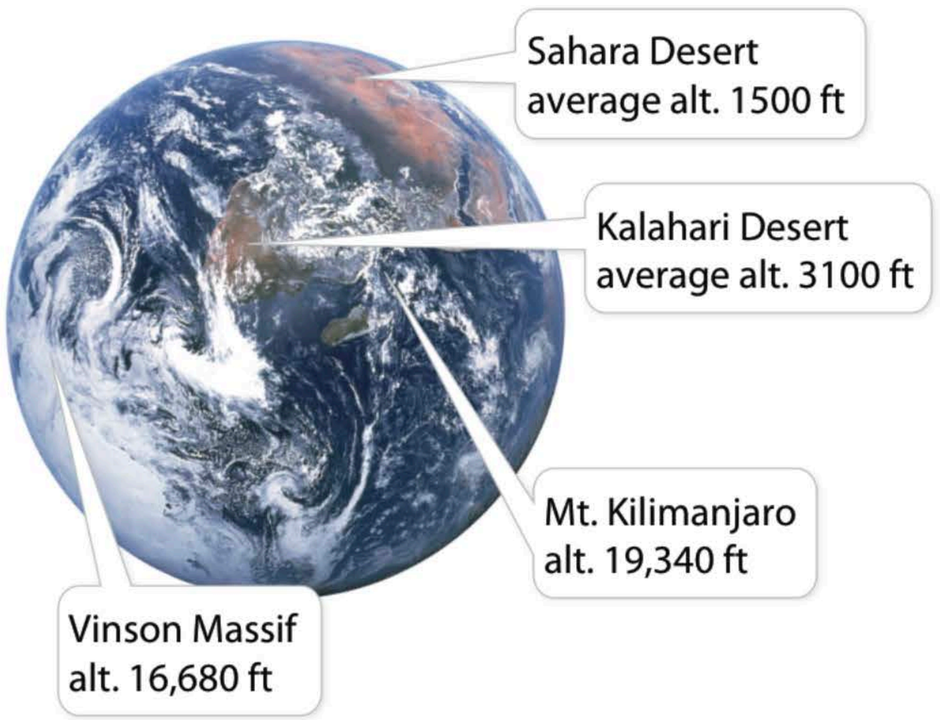
**36.** $10xy = -4$

**37.** **Writing** Explain how knowing the asymptotes of a translation of $y = \frac{1}{x}$ can help you graph the function. Include an example.

## Practice and Problem-Solving Exercises - *Continued*

**38. Multiple Choice** The formula $p = \frac{69.1}{a + 2.3}$ models the relationship between atmospheric pressure $p$ in inches of mercury and altitude $a$ in miles.

Use the data shown with the photo. At which location does the model predict the pressure to be about 23.93 in. of mercury? (*Hint:* 1 mi = 5280 ft.)




**A.** Sahara Desert

**B.** Kalahari Desert

**C.** Mt. Kilimanjaro

**D.** Vinson Massif

**Graphing Calculator** **Graph each pair of functions. Find the approximate point(s) of intersection.**

**39.** $y = \frac{6}{x - 2}$, $y = 6$

**40.** $y = -\frac{1}{x - 3} - 6$, $y = 6.2$

**41.** $y = \frac{3}{x + 1}$, $y = -4$

**42. Reasoning** How will the domain and the range of the parent function $y = \frac{1}{x}$ change after the translation of its graph by 3 units up and by 5 units to the left?

**43. a. Gasoline Mileage** Suppose you drive an average of 10,000 miles each year. Your gasoline mileage (mi/gal) varies inversely with the number of gallons of gasoline you use each year. Write and graph a model for your average mileage $m$ in terms of the gallons $g$ of gasoline used.

**b.** After you begin driving on the highway more often, you use 50 gal less per year. Write and graph a new model to include this information.

**c.** Calculate your old and new mileage assuming that you originally used 400 gal of gasoline per year.

## Practice and Problem-Solving Exercises - *Continued*

### C • Challenge

**Reasoning** **Compare each pair of graphs and find any points of intersection.**

**44.** $y = \frac{1}{x}$ and $y = \left|\frac{1}{x}\right|$

**45.** $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$

**46.** $y = \left|\frac{1}{x}\right|$ and $y = \frac{1}{x^2}$

**47.** Find two reciprocal functions such that the minimum distance from the origin to the graph of each function is $4\sqrt{2}$.

**48.** Write each equation in the form $y = \frac{k}{x-b} + c$, and sketch the graph.

**a.** $y = \frac{2}{3x-6}$

**b.** $y = \frac{1}{2-4x}$

**c.** $y = \frac{3-x}{x+2}$

**d.** $xy - y = 1$