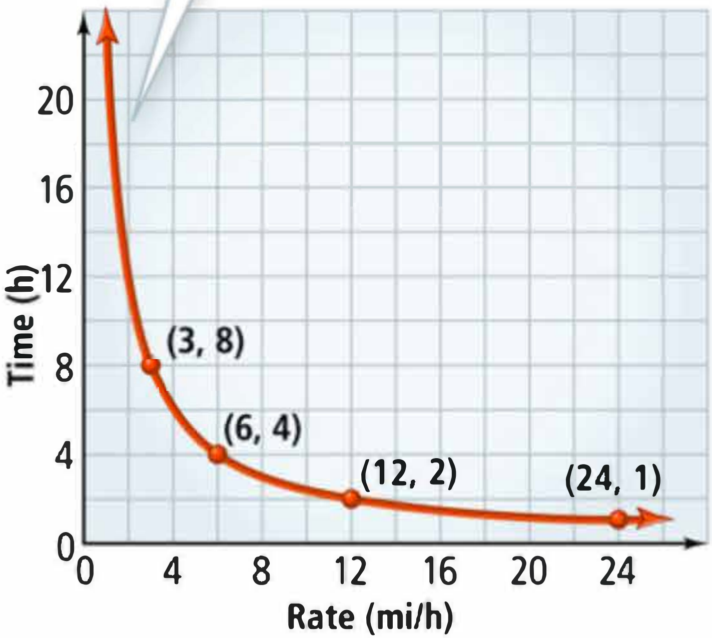
Among all rectangles with a given area, the longer the length of one side, the shorter the length of an adjacent side. If a product is constant, where the constant is positive, a decrease in the value of one factor must accompany an increase in the value of the other factor.

As an equation, direct variation has the form $y = kx$, where $k \neq 0$. **Inverse variation** can have the form $xy = k$, $y = \frac{k}{x}$, or $x = \frac{k}{y}$, where $k \neq 0$. When two quantities vary inversely, as one quantity increases, the other decreases proportionally.

For both inverse and direct variation, *k* is the constant of variation. The graph and the table below both show the time needed to bike 24 miles pedaling at different rates, *r*.

The inverse variation $t = \frac{24}{r}$ models this situation.



You can also see the inverse relationship, $t = \frac{24}{r}$ in this table. Notice that the product of the rate and time is always 24. The constant of variation is 24.

Rate (mi/h)	Time (h)
3	8
6	4
12	2
24	1

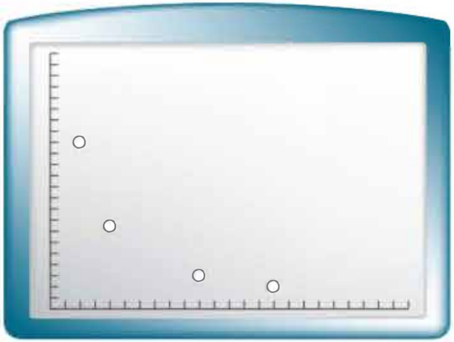
Problem 1 Identifying Direct and Inverse Variations

Is the relationship between the variables a *direct variation*, an *inverse variation*, or *neither*? Write function models for the direct and inverse variations.

A

x	y
2	15
4	7.5
10	3
15	2

As x increases, y decreases. This might be an inverse relationship. A plot confirms that an inverse relationship is possible. Test to see whether xy is constant.



$2 \cdot 15 = 30$ $\quad$ $4 \cdot 7.5 = 30$

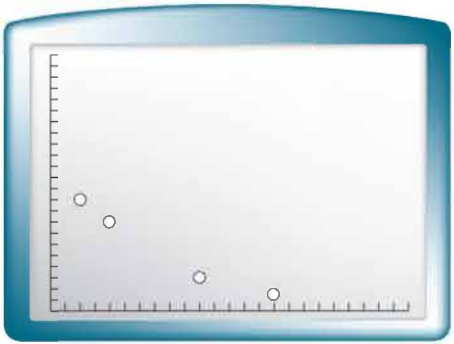
$10 \cdot 3 = 30$ $\quad$ $15 \cdot 2 = 30$

The product of each pair is 30, so $xy = 30$ and y varies inversely with x. The constant of variation is 30 and the function model is $y = \frac{30}{x}$.

B

x	y
2	10
4	8
10	3
15	1.5

A plot of the points suggests that an inverse relationship is possible. Test to see whether the products of x and y are constant.



$2 \cdot 10 = 20$, $4 \cdot 8 = 32$,

$10 \cdot 3 = 30$, and $15 \cdot 1.5 = 22.5$

Since the products are not constant, the relationship is not an inverse variation.

Problem 2 Determining an Inverse Variation

Suppose x and y vary inversely, and $x = 4$ when $y = 12$.

A What function models the inverse variation?

$y = \frac{k}{4}$ Write the general function form for inverse variation.

$12 = \frac{k}{4}$ Substitute for x and y.

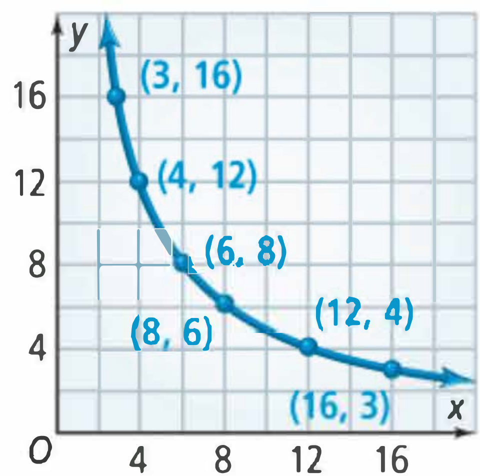
$k = 48$ Solve for k.

The function is $y = \frac{48}{x}$.

B What does the graph of this function look like?

Make a table of values. Sketch a graph.

x	y
3	16
4	12
6	8
8	6
12	4
16	3



C What is y when $x = 10$?

$$y = \frac{48}{x}$$ Write the function.

$$y = \frac{48}{10}$$ Substitute 10 for x.

$$y = 4.8$$ Simplify.

When $x = 10$, $y = 4.8$.

Problem 3 Modeling an Inverse Variation

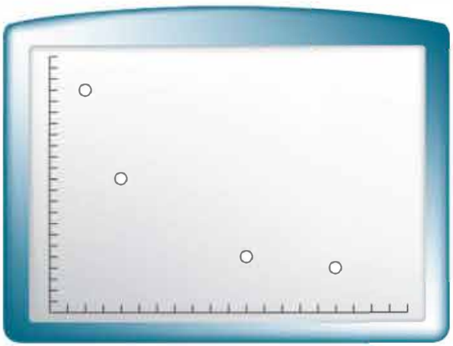
Your math class has decided to pick up litter each weekend in a local park. Each week there is approximately the same amount of litter. The table shows the number of students who worked each of the first four weeks of the project and the time needed for the pickup.

Park Cleanup Project

Number of students (n)	3	5	12	17
Time in minutes (t)	85	51	21	15

A What function models the data?

Step 1 Investigate the data. The more students who help, the less time the cleanup takes. An inverse variation seems appropriate. If this is an inverse variation, then $nt = k$. From the table, nt (or **L1 • L2**) is almost always 255.



L1	L2	L3 3
3	85	255
5	51	255
12	21	252
17	15	255

L3=L1 L2

Step 2 Determine the model.

$nt = 255$

B **How many students should there be to complete the project in at most 30 minutes each week?**

$$nt = 255$$ Use the model from part A.

$$n(30) = 255$$ Substitute for t.

$$nt = \frac{255}{30}$$ Solve for n.

$$= 8.5$$

There should be at least 9 students to do the job in at most 30 minutes.

You have seen many variation formulas in geometry. Some, like the formula for the perimeter of a square, are simple direct variations. Others, like the volume of a cone, relate three or more variables.

When one quantity varies with respect to two or more quantities, you have a **combined variation**. When one quantity varies directly with two or more quantities, you have **joint variation**. The volume of a cone varies jointly with the area of the base and the height of the cone, $V = kBh$.

TAKE NOTE Key Concept

Combined Variations

Combined Variation	Equation Form
z varies jointly with x and y.	$z = kxy$
z varies jointly with x and y and inversely with w.	$z = \frac{kxy}{w}$
z varies directly with x and inversely with the product wy.	$z = \frac{kx}{wy}$

Problem 4 Using Combined Variation

Multiple Choice **The number of bags of grass seed *n* needed to reseed a yard varies directly with the area *a* to be seeded and inversely with the weight *w* of a bag of seed. If it takes two 3-lb bags to seed an area of 3600 ft^2, how many 3-lb bags will seed 9000 ft^2?**

A. 3 bags **B.** 4 bags **C.** 5 bags **D.** 6 bags

$n = \frac{ka}{w}$ — *n* varies directly with *a* and inversely with *w*.

$2 = \frac{3600k}{3}$ — Substitute for *n*, *a*, and *w*.

$\frac{(2)(3)}{3600} = k$ — Solve for *k*.

$k = \frac{6}{3600} = \frac{1}{600}$ — Simplify.

The combined variation equation is $n = \frac{a}{600w}$.

$n = \frac{a}{600w}$ — Use the combined variation equation.

$= \frac{9000}{600(3)}$ — Substitute for *a* and *w*.

$= 5$ — Simplify.

You need five 3-lb bags to seed 9000 ft^2. The correct choice is C.

Problem 5 Applying Combined Variation

Physics Gravitational potential energy *PE* is a measure of energy. *PE* varies directly with an object's mass *m* and its height *h* in meters above the ground. Physicists use *g* to represent the constant of variation, which is gravity.

The skateboarder in the photo has a mass of 58 kg and a potential energy of 2273.6 joules. What is the gravitational potential energy of a 65-kg skateboarder on the halfpipe shown?

KNOW	NEED	PLAN
• The mass of each skateboarder • The height of each skateboarder • The potential energy of the first skateboarder	The potential energy of the second skateboarder	• Write the variation for potential energy. • Use the known information to find g. • Then find the potential energy of the second skateboarder.

Step 1 Write the formula for potential energy. Potential energy varies directly with mass and height. $PE = gmh$

Step 2 Use the given data to find g.

$PE = gmh$	Potential energy formula
$\frac{PE}{mh} = g$	Solve for g.
$\frac{2273.6}{(58)(4)} = g$	Substitute 58 for m and 4 for h.
$9.8 = g$	Simplify.

Step 3 Use the formula to find the potential energy of the second skateboarder.

$PE = 9.8mh$	Potential energy formula
$= 9.8(65)(4)$	Substitute 65 for m and 4 for h.
$= 2548$	Simplify.

The second skateboarder has 2548 joules of potential energy.

Lesson Check

Do you know HOW?

Is the relationship between the variables in each table a *direct variation*, an *inverse variation*, or *neither*? Write equations to model the direct and inverse variations.

1.

x	y
1	6
3	2
12	0.5
15	0.4

2.

u	v
−3	−15
5	25
6	30
16	80

Do you UNDERSTAND?

3. Compare and Contrast Describe the difference between direct variation and inverse variation.

4. Writing Describe how the variables in the given equation are related.

$$p = \frac{kqrt}{s}$$

5. Error Analysis A student described the relationship between the variables in the equation below as d varies directly with r and inversely with t. Correct the error in relating the variables.

$$d = \frac{k\sqrt[3]{r}}{t^2}$$

Practice and Problem-Solving Exercises

A • Practice

Is the relationship between the values in each table a *direct variation*, an *inverse variation*, or *neither*? Write equations to model the direct and inverse variations. SEE PROBLEM 1.

6.

x	y
3	15
8	40
10	50
22	110

7.

x	y
3	14
5	8.4
7	6
10.5	4

8.

x	y
0.5	1
2.1	4.2
3.5	7
11	22

9.

x	y
0.1	3
3	0.1
6	0.05
24	0.0125

Suppose that x and y vary inversely. Write a function that models each inverse variation. Graph the function and find y when $x = 10$. SEE PROBLEM 2.

10. $x = 1$ when $y = 11$

11. $x = -13$ when $y = 100$

12. $x = 1$ when $y = 1$

13. $x = 1$ when $y = 5$

14. $x = 1.2$ when $y = 3$

15. $x = 2.5$ when $y = 100$

16. $x = 20$ when $y = -4$

17. $x = 5$ when $y = -\frac{1}{3}$

18. $x = -\frac{4}{15}$ when $y = -105$

19. **Fundraising** In a bake sale, you recorded the number of muffins sold and the amount of sales in a table as shown. SEE PROBLEM 3.

Number of muffins (m)	Sales (s)
5	$12.50
8	$20.00
13	$32.50
20	$50.00

a. What is a function that relates the sales and the number of muffins?

b. How many muffins would you have to sell to make at least $250.00 in sales?

20. **Painting** The number of buckets of paint n needed to paint a fence varies directly with the total area a of the fence and inversely with the amount of paint p in a bucket. It takes three 1-gallon buckets of paint to paint 72 ft^2 of fence. How many 1-gallon buckets will be needed to paint 90 ft^2 of fence?

SEE PROBLEM 4.

21. **Potential Energy** On Earth with a gravitational acceleration g, the potential energy stored in an object varies directly with its mass m and its vertical height h. SEE PROBLEM 5.

a. What is an equation that models the potential energy of a 2-kg skateboard that is sliding down a ramp?

b. The acceleration due to gravity is $g = -9.8\ \text{m/s}^2$? What is the height of the ramp if the skateboard has a potential energy of $-39.2\ \text{kg m}^2/\text{s}^2$?

B • Apply

22. Think About a Plan The table shows data about how the life span s of a mammal relates to its heart rate r. The data could be modeled by an equation of the form $rs = k$. Estimate the life span of a cat with a heart rate of 126 beats/min.

Heart Rate and Life Span

Mammal	Heart rate (beats/min)	Life span (min)
Mouse	634	1,576,800
Rabbit	158	6,307,200
Lion	76	13,140,000

SOURCE: *The Handy Science Answer Book*

- How can you estimate a constant of the inverse variation?
- What expression would you use to find the life span?

23. Physics The force F of gravity on a rocket varies directly with its mass m and inversely with the square of its distance d from Earth. Write a model for this combined variation. Write an equation to find the mass of the rocket in terms of F and d.

24. The spreadsheet shows data that could be modeled by an equation of the form $PV = k$. Estimate P when $V = 62$.

	A	B
1	P	V
2	140.00	100
3	147.30	95
4	155.60	90
5	164.70	85
6	175.00	80
7	186.70	75

25. **Chemistry** The formula for the Ideal Gas Law is $PV = nRT$, where P is the pressure in kilopascals (kPA), V is the volume in liters (L), T is the temperature in Kelvin (K), n is the number of moles of gas, and $R = 8.314$ is the universal gas constant.

a. Write an equation to find the volume in terms of P, n, R, and T.

b. What volume is needed to store 5 moles of helium gas at 350 K under the pressure 190 kPA?

c. A 10 L cylinder is filled with hydrogen gas to a pressure of 5,000 kPA. The temperature of gas is 300 K. How many moles of hydrogen gas are in the cylinder?

Write the function that models each variation. Find z when $x = 4$ and $y = 9$.

26. z varies directly with x and inversely with y. When $x = 6$ and $y = 2$, $z = 15$.

27. z varies jointly with x and y. When $x = 2$ and $y = 3$, $z = 60$.

28. z varies inversely with the product of x and y. When $x = 2$ and $y = 4$, $z = 0.5$.

Each pair of values is from a direct variation. Find the missing value.

29. $(2, 5)$, $(4, y)$

30. $(4, 6)$, $(x, 3)$

31. $(3, 7)$, $(8, y)$

32. $(x, 12)$, $(4, 1.5)$

Each ordered pair is from an inverse variation. Find the constant of variation.

33. $(6, 3)$

34. $(0.9, 4)$

35. $\left(\frac{3}{8}, \frac{2}{3}\right)$

36. $\left(\sqrt{2}, \sqrt{18}\right)$

Each pair of values is from an inverse variation. Find the missing value.

37. $(2, 5), (4, y)$

38. $(4, 6), (x, 3)$

39. $(3, 7), (8, y)$

40. $(x, 12), (4, 1.5)$

C • Challenge

41. Writing Explain why 0 cannot be in the domain of an inverse variation.

42. Reasoning Suppose that (x_1, y_1) and (x_2, y_2) are values from an inverse variation. Show that $\frac{x_1}{x_2} = \frac{y_2}{y_1}$.

43. Open-Ended The height h of a cylinder varies directly with its volume V and inversely with the square of its radius r. Find at least four ways to change the volume and radius of a cylinder so that its height is quadrupled.

Standardized Test Prep

SAT/ACT

44. Which equation represents inverse variation between x and y?

A. $x = \frac{y}{z}$

B. $x = \frac{-15z}{y}$

C. $z = \frac{-15y}{x}$

D. $xz = 5y$

Solve each equation. Check your answers.
SEE LESSON 7-6.

49. $\ln 4 + \ln x = 5$

50. $\ln x - \ln 3 = 4$

51. $2\ln x + 3\ln 4 = 4$

Multiply and simplify. SEE LESSON 6-2.

52. $-5\sqrt{6x} \cdot 3\sqrt{6x^3}$

53. $3\sqrt[3]{2x^2} \cdot 7\sqrt[3]{32x^4}$

54. $\sqrt{5x^3} \cdot \sqrt{40xy^7}$

Get Ready To prepare for Lesson 8-2, do Exercises 55–60.

Graph each equation. Then describe the transformation of the parent function $f(x) = |x|$.

SEE LESSON 2-7.

55. $y = |x| + 2$

56. $y = |x + 2|$

57. $y = |x| - 3$

58. $y = |x - 3|$

59. $y = |x + 4| - 5$

60. $y = |x - 10| + 7$