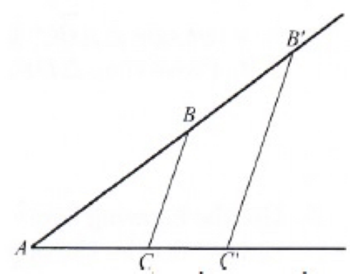
MATH 7
ASSIGNMENT 23: EUCLIDEAN GEOMETRY IV
APR 19, 2020

1. **Similar triangles**

We say that triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar with coefficient k if $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $\angle C = \angle C'$ and
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = k.$$
We will use notation $\triangle ABC \sim \triangle A'B'C'$.

Theorem 13. *Consider a triangle $\triangle ABC$ and let $B' \in \overrightarrow{AB}$, $C' \in \overrightarrow{AC}$ be such that lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel. Then $\triangle ABC \sim \triangle A'B'C'$.*



Theorem 14. *For any triangle $\triangle ABC$ and a real number $k > 0$, there exists a triangle $\triangle A'B'C'$ similar to $\triangle ABC$ with coefficient k.*

Theorem 15 (Similarity via (AA))**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that $m\angle A = m\angle A'$, $m\angle B = m\angle B'$. Then these triangles are similar.*

Proof. Let $k = \frac{A'B'}{AB}$. Construct a triangle $\triangle A''B''C''$ which is similar to $\triangle ABC$ with coefficient k. Then $A'B' = A''B''$, and $m\angle A = m\angle A' = m\angle A''$, $m\angle B = m\angle B' = m\angle B''$. Thus, by (ASA), $\triangle A'B'C' \cong \triangle A''B''C''$. □

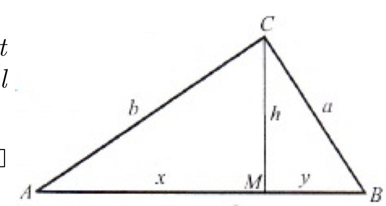
Theorem 16 (Similarity via (SAS))**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that $\angle A = \angle A'$, $\frac{A'B'}{AB} = \frac{A'C'}{AC}$. Then these triangles are similar.*

Theorem 17 (Similarity via (SSS))**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that*
$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$$
Then these triangles are similar.

One of the most important applications of the theory of similar triangles is to the study of right triangles and the Pythagorean theorem. A *right* triangle is a triangle in which one of the angles is a right angle. A *hypotenuse* is the side opposing the right angle; the two other sides are called legs.

Theorem 18. *Let $\triangle ABC$ be a right triangle, with $\angle C$ being the right angle. Let CM be the altitude of angle C. Then triangles $\triangle ABC$, $\triangle ACM$, $\triangle CBM$ are all similar.*

Proof. It immediately follows from (AA) similarity rule. □



This theorem immediately implies a number of important relations between various lengths in these triangles. We will give one of them. Denote for brevity $a = BC$, $b = AC$, $c = AB$, $x = AM$, $y = MB$, $h = CM$. Then we have $x : h = b : a$, $y : h = a : b$, so
$$\frac{x}{h} \times \frac{y}{h} = 1$$
or $xy = h^2$.

Homework

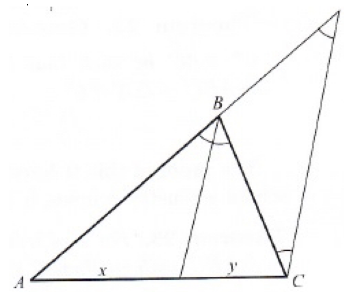
In the problems about constructing something with a ruler and compass, the ruler can only be used for drawing straight lines through two given points; you can not use it to measure distances. You can freely use previous results and constructions without repeating all the steps.

1. Prove theorem 16.

2. Use the drawing of the rectangular triangle in the previous page to prove that:
(a) $cx = b^2$
(b) $cy = a^2$
(c) Prove Pithagoras' theorem: $a^2 + b^2 = c^2$

3. In a triangle $\triangle ABC$, let D be midpoint of side BC, E – midpoint of side AC, F – midpoint of side AB. Prove that $\triangle DEF$ is similar to triangle $\triangle ABC$ with coefficient 1/2.

4. Use the folowing figure to prove that an angle bisector in a triangle $\triangle ABC$ divides the opposite side in the same proportion as the two adjoining sides: $\frac{x}{y} = \frac{BA}{BC}$.



5. Given an angle $\angle POQ$ and a point M inside it, construct points A, B on the sides of this angle so that AB goes through M and $\triangle AOB$ is isosceles. [Hint: first construct any isosceles triangle with vertices on the sides of the given angle; then the required triangle must be similar to it.]

Extra Problems

6. Let $ABCD$ be a trapezoid with bases $AD = 9$, $BC = 6$, such that the height (distance between the bases) is equal to 5. Let O be the intersection point of lines AB, CD.
(a) Show that triangles $\triangle OBC$, $\triangle OAD$ are similar and find the coefficient.
(b) Find the distance from O to AD (i.e., length of the perpendicular).

7. Given three line segments, of lenghts $1, x, y$, construct the line segments of lenghts xy, x/y, using only ruler and compass.