MATH 7
ASSIGNMENT 21: EUCLIDEAN GEOMETRY II
MAR 29, 2020

1. Parallel and perpendicular lines

Theorem 6. *Given a line l and point P not on l, there exists exactly one line m through P which is parallel to l.*

Proof. **Existence:** Let us draw a line k through P which intersects l. Now draw a line m through P such that alternate interior angles are equal: $m\angle 1 = m\angle 2$. Then, by Axiom 4 (alternate interior angles), we have $m \parallel l$.
Uniqueness: To show that such a line is unique, let us assume that there are two different lines, m_1, m_2 through P both parallel to l. By Theorem 2, this would imply $m_1 \parallel m_2$. This gives a contradiction, because they both go through P(Figure 1).

□

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

2. Sum of angles of a triangle

Definition 1. A triangle is a figure consisting of three distinct points A, B, C (called vertices) and line segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$. We denote such a triangle by $\triangle ABC$.

Similarly, a quadrilateral is a figure consisting of 4 distinct points A, B, C, D and line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ such that these segments do not intersect except at A, B, C, D.

Theorem 8. *The sum of measures of angles of a triangle is* $180°$.

Proof. Draw a line m through B parallel to $\overleftrightarrow{AC}$ (possible by Theorem 6). Let D, E be points on m as shown in the Figure 2.

Then $m\angle DBA = m\angle A$ as alternate interior angles, $m\angle CBE = m\angle C$. On the other hand, by Axiom 3 (angles add up), we have

$$m\angle DBA + m\angle B + m\angle CBE = 180°$$

Thus, $m\angle A + m\angle B + m\angle C = 180°$. □

Theorem 9. *For a triangle $\triangle ABC$, let D be a point on continuation of side AC, so that C is between A and D. Then $m\angle BCD = m\angle A + m\angle B$. (Such an angle is called the exterior angle of triangle ABC.)*

Theorem 10. *Sum of angles of a quadrilateral is equal to* $360°$.

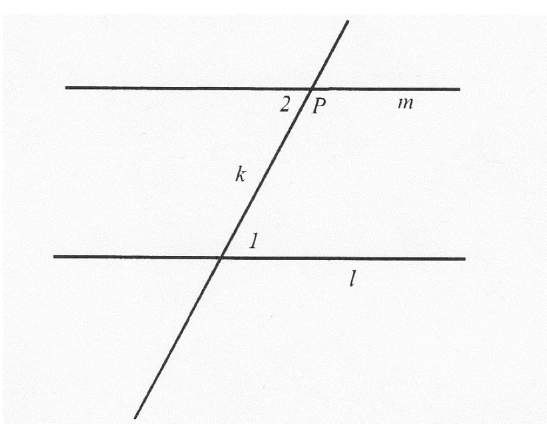


FIGURE 1. Parallel Lines

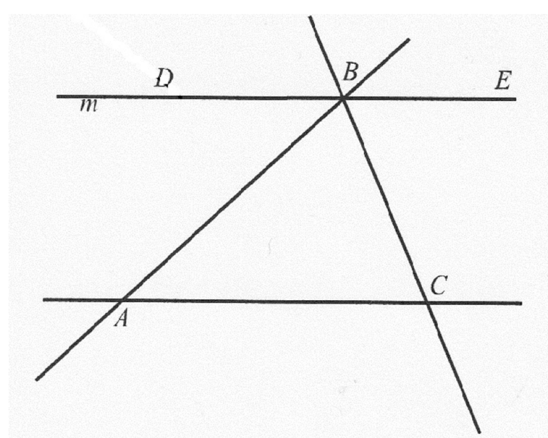
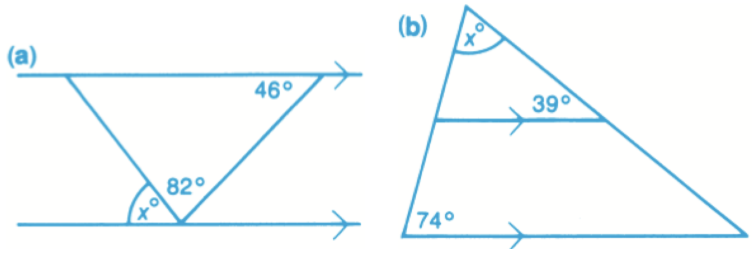


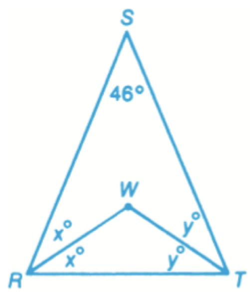
FIGURE 2. Sum of Angles

Homework

1. In each of the following pictures find the value of x:



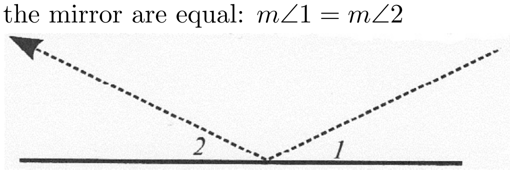
2. Find the measure of angle $\angle RWT$:



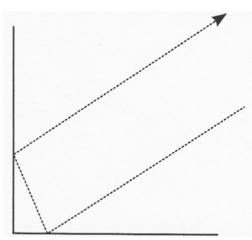
3. Prove Theorem 7.

4. Prove Theorem 9.

5. The reflection law states that the angles formed by the incoming light ray and the reflected one with the surface of the mirror are equal: $m\angle 1 = m\angle 2$



Using this law, show that a corner made of two perpendicular mirrors will reflect any light ray exactly back: the reflected ray is parallel to the incoming one:

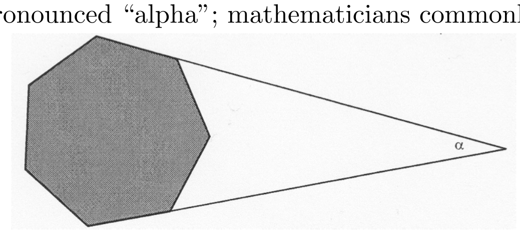


This property – or rather, similar property of corners in 3-D – is widely used: reflecting road signs, tail lights of a car, reflecting strips on clothing are all constructed out of many small reflecting corners so that they reflect the light of a car headlamp exactly back to the car.

Extra Problems

6. Deduce a formula for the sum of angles in a polygon with n vertices.

7. In the figure below, all angles of the 7-gon are equal. What is angle α?[By the way: α is a Greek letter, pronounced "alpha"; mathematicians commonly use Greek letters to denote angles]



8. Show that if, in a quadrilateral $ABCD$, diagonally opposite angles are equal ($m\angle A = m\angle C$, $m\angle B = m\angle D$), then opposite sides are parallel. [Hint: show first that $m\angle A + m\angle B = 180°$.]