MATH 7
ASSIGNMENT 20: EUCLIDEAN GEOMETRY
MAR 21, 2020

Today we start the study of Euclidean geometry. When studying Euclidean geometry, we focus heavily on proofs. We try to start from some very simple "assumptions", called Axioms, for example by defining simple properties of points and lines. Then everything else should be proved starting from those assumptions. It will be useful to review your material on logic from last year.

One very useful result from logic is: if A implies B then, whenever B is false, A is necessarily false. This is used in proving by contradiction: suppose we want to prove a certain statement. If we can show that assuming that this statement is false leads to a contradiction, then we are done.

Basic objects

These objects are the basis of all our constructions: all objects we will be discussing will be defined in terms of these objects. No definition is given for these basic objects.

- Points
- Lines
- Distances: for any two points A, B, there is a non-negative number AB, called distance between A, B.
- Angle measures: for any angle $\angle ABC$, there is a real number $m\angle ABC$, called the measure of this angle (more on this later).

We will also frequently use words "between" when describing relative position of points on a line (as in: A is between B and C) and "inside" (as in: point C is inside angle $\angle AOB$).

Having these basic notions, we can now define more objects. Namely, we can give definitions of

- an interval, or line segment is a part of a line consisting of two points, called end points, and the set of all points between them. ($\overline{AB}$)
- a ray is a part of a line consisting of a given point, called the end point, and the set of all points on one side of the end point.
- an angle is the union of two rays having the same end point. The end point is called the vertex of the angle, and the rays are called the sides of the angle. (notation: $\angle AOD$)
- parallel lines: two distinct lines l, m are called parallel (notation: $l \parallel m$) if they do not intersect, i.e. have no common points

First postulates

Axiom 1. *For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

Axiom 2. *If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$. (see Figure 1)*

Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in Figure 2 below (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*

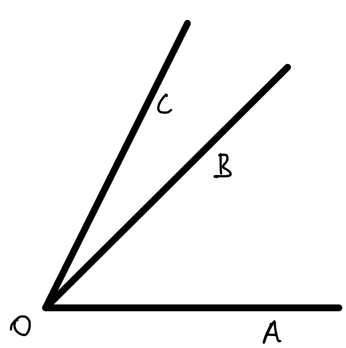


FIGURE 1. Angle Addition

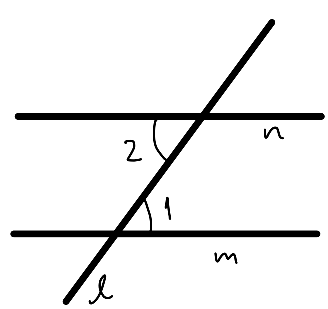


FIGURE 2. Alt. Int. Angles

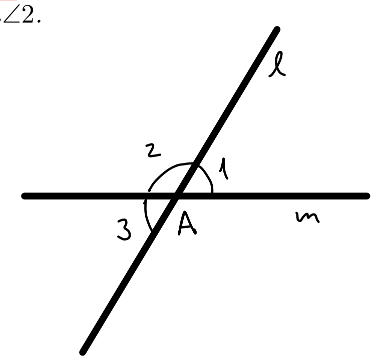


FIGURE 3. Vertical Angles

First theorems

Theorem 1. *If lines l, m intersect, than they intersect at exactly one point.*

Proof. Assume that they intersect at more than one point. Let P, Q be two of the points where they intersect. Then both l, m go through P, Q. This contradicts Axiom 1. Thus, our assumption (that l, m intersect at more then one point) must be false. □

Theorem 2. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Theorem 3. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in figure 3 (such a pair of angles are called* vertical*). Then $m\angle 1 = m\angle 3$.*

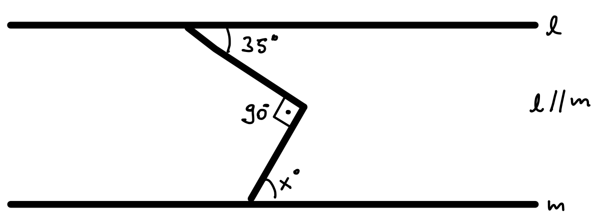
Proof. Let angle 2 be as shown in the Figure 3 . Then, by Axiom 3, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. □

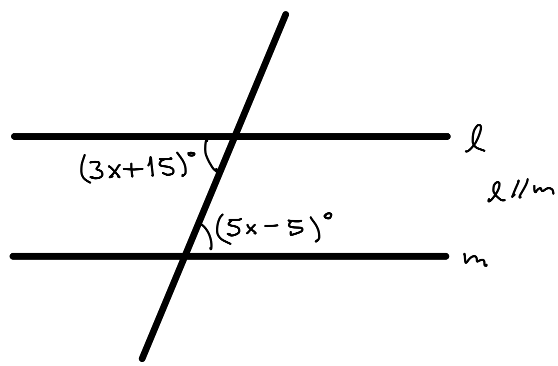
Theorem 4. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are* perpendicular *and write $l \perp m$.)*

Theorem 5. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

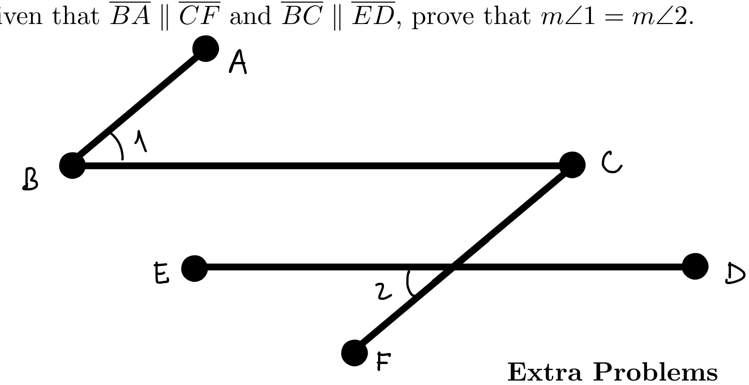
Homework

1. Prove Theorem 4.
2. Prove Theorem 5.
3. In each of the following pictures find the value of x:





4. Given that $\overline{BA} \parallel \overline{CF}$ and $\overline{BC} \parallel \overline{ED}$, prove that $m\angle 1 = m\angle 2$.



Extra Problems

5. Prove Theorem 2. [*Hint:* assume that l and n are not parallel; then they must intersect at some point P...]
6. Suppose we draw k lines on the plane so that each of them intersects each other, and all intersection points are distinct. Into how many pieces will they cut the plane? [Hint: how does the number of pieces change when you increase k by 1, i.e. add one more line?]
7. Suppose that instead of studying geometry on the plane, we study geometry on the sphere (say, Earth surface) and take lines to be equators, i.e. intersections of the sphere with a plane going through the center of the sphere. Which of the axioms will be true in this new, "spherical", geometry? Which will be false? Can you suggest a new set of axioms to describe this geometry?