MATH 7
ASSIGNMENT 19: DE MOIVRE'S FORMULAS

MAR 8, 2020

Product of Complex Numbers

As we saw in the previous class, the product of complex numbers corresponds, in polar form, to the product of the modulus and sum of the argument:

Theorem.

$$[\rho_1(\cos\theta_1 + i\sin\theta_1)][\rho_2(\cos\theta_2 + i\sin\theta_2)] = (\rho_1 \cdot \rho_2)(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)).$$

In order to prove this we used the formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{1}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \tag{2}$$

Today we briefly discussed how to prove these formulas geometrically.

De Moivre's 1st Formula

A direct consequence of the above formula is that powers become very simple in polar form.

$$[\rho(\cos\theta + i\sin\theta)]^n = [\rho^n(\cos n\theta + i\sin n\theta)]$$

We can check this with a few examples:

$$i^2 = [1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})]^2 = [1^2(\cos\pi + i\sin\pi)] = -1$$

$$(\sqrt{2}+i\sqrt{2})^4 = [2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})]^4 = [2^4(\cos\pi + i\sin\pi)] = -16$$

We can quickly check that we get the same result for the second example as when we use the algebraic form:

$$(\sqrt{2}+i\sqrt{2})^4 = [(\sqrt{2}+i\sqrt{2})^2]^2 = [(2-2)+i(2+2)]^2 = [4i]^2 = -16.$$

For large powers, it is clear that De Moivre's formula is more useful.

Combining De Moivre's formula and Pascal's triangle

We can use these two ways of expressing the multiplication of vectors to derive trigonometric identities. For example, what is the value of the cosine of 15° (equivalently, $\cos\frac{\pi}{12}$)?

Here is a way to find out. First, using De Moivre's formula,

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

On the other hand, using the third line of Pascal's triangle,

$$(\cos\theta + i\sin\theta)^2 = (\cos\theta)^2 + 2(\cos\theta)(i\sin\theta) + (i\sin\theta)^2 = [(\cos\theta)^2 - (\sin\theta)^2] + i[2(\sin\theta)(\cos\theta)]$$

Comparing these two, we find that

$$\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2$$

$$\sin 2\theta = 2(\sin\theta)(\cos\theta)$$

Taking $\theta = \frac{\pi}{12}$, we find the equations

$$\frac{\sqrt{3}}{2} = (\cos\frac{\pi}{12})^2 - (\cos\frac{\pi}{12})^2$$
$$\frac{1}{2} = 2(\sin\frac{\pi}{12})(\cos\frac{\pi}{12}).$$

Solving these (that's a challenge for you!) equations, we find that

$$\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}} \approx 0.97.$$

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&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

De Moivre's 2nd Formula

A similar form holds for the n-th roots of a polynomial,

$$\sqrt[n]{\rho(\cos\theta + i\sin\theta)} = \sqrt[n]{\rho}[\cos(\frac{\theta}{n} + \frac{2\pi}{n}k) + i\sin(\frac{\theta}{n} + \frac{2\pi}{n}k)].$$

For any integer k, this formula gives an n-th root. But in order to list all the different ones it is enough to consider the cases $k \in \{0, 1, 2, ..., n-2, n-1\}$. This way this formula gives all the n distinct n-th roots of a complex number.

There is a beautiful geometric picture that goes with this formula: since all the n-th roots of a complex number have the same modulus and their arguments differ by multiples of $\frac{2\pi}{n}$, they form the vertices of a *regular n-gon.*

Example. Let us find the cubic roots of 8. We know that $2^3 = 8$ but by the above discussion, there are two other complex numbers that are also cubic roots. Indeed,

$$\sqrt[3]{8} = \sqrt[3]{8[\cos 0 + i\sin 0]} = 2[\cos(\frac{2\pi}{3}k) + i\sin(\frac{2\pi}{3}k)].$$

For $k = 0$, we get 2. For $k = 1$, we get

$$2[\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})] = 2[-\frac{1}{2} + i\frac{\sqrt{3}}{2}] = -1 + i\sqrt{3}.$$

And, for $k = 1$, we find the last root,

$$2[\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3})] = 2[-\frac{1}{2} - i\frac{\sqrt{3}}{2}] = -1 - i\sqrt{3}.$$

These three cubic roots of 8 form an equilateral triangle in the complex plane.

Homework

1. Evaluate (you can leave the number in polar form)
- (a) $[3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})]^4$
- (b) $[2(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})]^2$
- (c) $[\frac{1}{3}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})]^2$
- (d) $[\sqrt{2}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})]^4$

2. In each case, evaluate (you don't need to evaluate big numbers like 3^{20})
- (a) $(3-3i)^{-12}$
- (b) $(\sqrt{2} + i\sqrt{2})^8$
- (c) $(-1+i)^6$
- (d) $(1-i\sqrt{3})^{-5}$

[Hint: many of these cases are easier to calculate in polar form.]

3. Find and draw on the complex plane the
- (a) Square-roots of -1

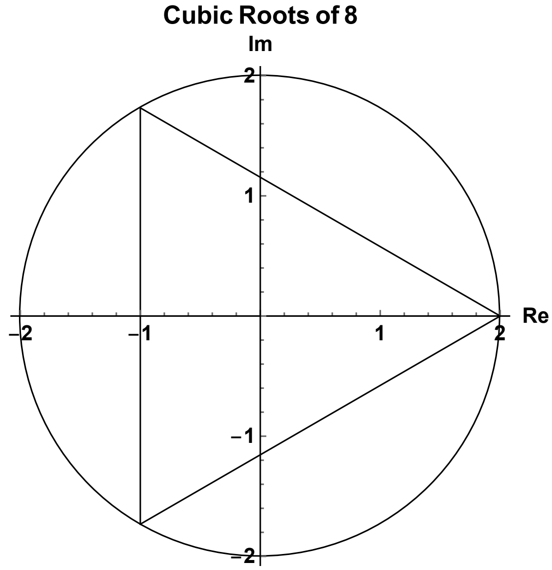

FIGURE 1. The three cubic roots of 8 form an equilateral triangle in the complex plane.

(b) Fourth-roots of 1
(c) Cubic-roots of 27
(d) Fourth-roots of 16
(e) Fourth-roots of -16

[Hint: many of these cases are easier to calculate in polar form.]

4. Find all the complex roots
(a) $\sqrt[3]{1+i}$
(b) $\sqrt{-16i}$
(c) $\sqrt{5+12i}$
(d) $\sqrt{\frac{-1+i\sqrt{3}}{2}}$

5. A square inscribed in a circle with center at 0 has $3i$ as one of its vertices. Find the complex numbers corresponding to the other three vertices.

Additional Problems (Optional)

1. Draw (no need to calculate) all the values of $\sqrt[6]{64}$

2. What is the smallest positive integer n for which the number $(i-\sqrt{3})^n$ is purely imaginary (the real part is zero)?

3. Use De Moivre's formula and Pascal's triangle to express $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$. [Hint: Use a strategy similar to the one we used in class for the $\cos 2\theta$.]