MATH 7
ASSIGNMENT 18: POLAR REPRESENTATION OF COMPLEX NUMBERS

MAR 1, 2020

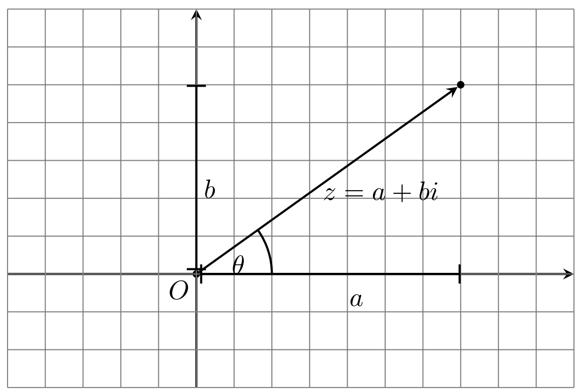
Complex Numbers

Recall that any complex number z can be written as

$$z = a + bi,$$

where a and b are real numbers. They are called, respectively, the real part of z ($a = Re(z)$ and the imaginary part of z ($b = Im(z)$).

We can represent complex numbers in the plane:



Addition of complex numbers is given by adding the real and the imaginary parts,

$$(a + bi) + (c + di) = (a + b) + (c + d)i.$$

The product is given by using the distributive property and then remembering that $i^2 = -1$,

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

Absolute value, Complex Conjugate and Inverse

The length of the vector corresponding to the complex number z is called the *absolute value* of z, $|z|$. Clearly,

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

It is convenient to also introduce the *complex conjugate* $\bar{z}$,

$$\bar{z} = \overline{(a + bi)} = a - bi.$$

Then we have the simple expression $|z|^2 = z\bar{z}$:

$$z\bar{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2 = |z|^2.$$

We can also use these definitions to write down the inverse of a complex number as

$$z^{-1} = \frac{\bar{z}}{|z|^2}.$$

Indeed,

$$z\left(\frac{\bar{z}}{|z|^2}\right) = \frac{|z|^2}{|z|^2} = 1.$$

Polar form

Let θ be the angle that z makes with the x-axis (see figure above). It is called the *argument* of z. Then, any complex number $z = x + yi$ can be written in the polar form

$$z = \rho(\cos\theta + i\sin\theta),$$

where

$$\rho = |z|, \;\; \cos\theta = \frac{x}{\rho}, \;\; \sin\theta = \frac{y}{\rho}.$$

For example,

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + \sin\frac{\pi}{3}.\right)$$

Geometrical Interpretation of the Product

We saw that the addition of complex numbers has simple geometric interpretations, but until now we don't know of an analogously geometrical definition of the product of two complex numbers. If the numbers are written in polar form, this interpretation becomes clear:

Theorem.

$$[\rho_1(\cos\theta_1 + i\sin\theta_1)][\rho_2(\cos\theta_2 + i\sin\theta_2)] = (\rho_1 \cdot \rho_2)(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

To show this we need the following formulas:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{1}$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \tag{2}$$

Indeed, one can check that

$$[\rho_1(\cos\theta_1 + i\sin\theta_1)][\rho_2(\cos\theta_2 + i\sin\theta_2)] = (\rho_1\rho_2)(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1).$$

We see that the geometric interpretation is that $z_1 z_2$ is a complex number whose size is the product of the sizes of z_1 and z_2 and whose argument is the sum of the arguments of z_1 and z_2.

Homework

1. Prove that, for any complex number $z = a + ib$,

$$Re(z) = \frac{z + \bar{z}}{2}, \;\; Im(z) = \frac{z - \bar{z}}{2i}.$$

2. In each case, simplify the given complex number in the form $a + bi$ for some a and b.

(a) $\frac{3}{2+i}$
(b) $\frac{1+2i}{3-i}$
(c) $\frac{2}{i}$
(d) $\frac{1+i}{(1-i)^2}$
(e) $\frac{1}{1-7i}$
(f) $\frac{3+4i}{2-i}$

[Hint: to simplify z_1/z_2, try multiplying by $\bar{z}_2/\bar{z}_2$]

3. Find a (real) such that the number

$$z = \frac{1+2i}{2+ai}$$

is real.

4. Find a complex number z such that

$$z(5 + 8i)$$

is real and

$$\frac{z}{1+i}$$

is purely imaginary (which means that it's real part is zero).

5. For each number, find its absolute value and argument.

(a) 4
(b) $1 + i\sqrt{3}$
(c) $3i$

(d) $-\sqrt{2}+i\sqrt{2}$
(e) -5
(f) $-2i$
(g) $-5-5i$
(h) $2-2i$

6. Sketch the position of these numbers in the complex plane, explicitly indicating the argument (angle with x-axis) (note that you can use your results from the previous problem).
(a) 4
(b) $1+i\sqrt{3}$
(c) $3i$
(d) $-\sqrt{2}+i\sqrt{2}$
(e) $-5-5i$

Additional Problems (Optional)

1. If z and w are complex numbers such that $z^2-w^2=6$ and $\bar{z}+\bar{w}=1-i$, then what is the value of $z-w$?
2. If $z+\frac{1}{z}=-1$, then what is the value of $|z|$?
3. Let $z=\rho(\cos\theta+i\sin\theta)$ and $w=2(\cos(\pi/6)+i\sin(\pi/6))$. Then the points w, $w+z$, $w+iz$, $w+z+iz$ are vertices of what shape?
4. Sketch the set of complex numbers z such that $|z-t|<1$, where $t=2+3i$.