MATH 7
ASSIGNMENT 17: COMPLEX NUMBERS
FEB 23, 2020

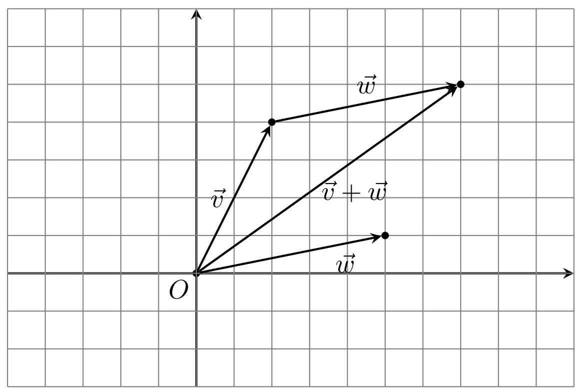
Vectors in Coordinates

Recall that a vector $\vec{v}$ in the plane can be expressed in terms of its two components $\vec{v} = (v_x, v_y)$.

In terms of the components, the sum of two vectors and the product by a real number become very simple:

$$\vec{v} + \vec{w} = (v_x + w_x, v_y + w_y),$$
$$t\vec{v} = (tv_x, tv_y),$$

as one can see from the picture:



Theorem. *These operations have some nice properties:*

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$
$$(\vec{v_1} + \vec{v_2}) + \vec{v_3} = \vec{v_1} + (\vec{v_2} + \vec{v_3})$$
$$t(\vec{v}_1 + \vec{v}_2) = t\vec{v}_1 + t\vec{v}_2$$

and so on.

When we are not thinking much about the vectors, but just about the pairs of real numbers of the form $(\vec{v}_1, \vec{v}_2)$, we call the set of these pairs $\mathbb{R}^2$.

The product in $\mathbb{R}^2$

As we discussed in the last class, there is no obvious way to multiply two vectors.

But it turns out that for vectors in the plane (but not, for example, for vectors in three-dimensional space) there is a way to multiply them! Define:

$$(a, b)(c, d) = (ac - bd, ad + bc).$$

Then this multiplication works quite a lot like the multiplication of real numbers. For example,

$$(a, b)(c, d) = (c, d)(a, b)$$
$$((a, b)(c, d))(e, f) = (a, b)((c, d)(e, f))$$
$$(a, b)((c, d) + (e, f)) = (a, b)(c, d) + (a, b)(e, f)$$

and so on.

The Complex Numbers

Let us try to understand how the operations above work for the simple case of vectors along the x-axis. Then, using the definitions above,

$$(a, 0) + (b, 0) = (a + b, 0)$$
$$(a, 0)(b, 0) = (ab, 0).$$

so they work just like the real numbers! We therefore call the more complex case (a, b) the *complex numbers*. The symbol for complex numbers is $\mathbb{C}$.

The Imaginary Unit

There is a better notation for complex numbers. Note that any complex number (a, b) can be written in the form

$$(a, b) = (a, 0)(1, 0) + (b, 0)(0, 1)$$

We already saw that $(a, 0)$, $(b, 0)$ and (1,0) correspond to the real numbers a, b and 1, respectively. We call $(0, 1)$ the *imaginary unit*, denoted by i. Thus we write the complex number (a, b) as

$$a + bi.$$

We know how to do the product of real numbers. How about i? It's easy to determine:

$$(0, 1)(0, 1) = (-1, 0) \quad \Rightarrow \quad i^2 = -1.$$

With this, we can summarize complex numbers in the following way:

Complex numbers are of the form $a + bi$, where a and b are real numbers and i is the imaginary unit. The sum of two complex numbers is

$$(a + bi) + (c + di) = (a + b) + (c + d)i$$

and the product is

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

Homework

1. Using the definition

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

calculate $z_1 + z_2$, $z_1 z_2$ and z_1^2 for
- $z_1 = (2, 1)$ and $z_2 = (1, 3)$
- $z_1 = (3, 4)$ and $z_2 = (1, -2)$

2. In each case, using the definition of product in the previous exercise, calculate $((a, b)(c, d))(e, f)$ and $(a, b)((c, d)(e, f))$. Are they equal?
- For $(a, b) = (1, 2)$, $(b, c) = (3, 2)$ and $(3, 1)$.
- For general complex numbers (a, b), (c, d) and (e, f).

3. Calculate the following:
- $(3 + 2i) + (2 - 5i)$
- $(1 + i) + (1 - i) - 2i$
- $(5 - 2i) - (2 + 8i)$
- $(6 + 7i) - (2 - 4i) + 10i$

4. Calculate the following:
- $(3 + 2i)(2 - 5i)$
- $(1 + i)(1 - i)(2i)$
- $(5 - 2i)(2 + 8i)$
- $(6 + 7i)(2 - 4i)(8i - 1)$
- $(1 + i)^3$

5. What is the value of i^n for n some positive integer?

6. In each case, find the real numbers x and y which satisfy the equation.
- $(3 + yi) + (x - 2i) = 7 - 5i$
- $2 + 3yi = x + 9i$
- $(x + yi)(2 + 3i) = 1 + 8i$
- $(x + yi)^2 = -9$
- $(x + yi)^2 = 4i$

Additional Problems (Optional)

1. Consider a complex number $z = a + bi$. Show that
(a) The length of the vector corresponding to z is $\sqrt{a^2 + b^2}$.
(b) The inverse $z^{-1} = \frac{a-bi}{a^2+b^2}$ is the inverse of z, ie. show that $zz^1 = 1$.

2. Express

$$z = \frac{(1 + i)^{80} - (1 + i)^{82}}{i^{96}}$$

in the form $a + bi$, where a and b are some real numbers.