MATH 7
ASSIGNMENT 14: BINOMIAL THEOREM AND PROBABILITIES

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Binomial Theorem

Binomial coefficients that appear in Pascal Triangle have an important application in algebra. They allow us to find an expansion of expressions such that $(a+b)^2, (a+b)^3, (a+b)^4, (a+b)^5, (a+b)^6$?

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n}b^n$$

Notice that the coefficients correspond to a row in Pascal's Triangle. The coefficient of the term $a^{n-k}b^k$ is $\binom{n}{k}$.

Square of a Sum, Difference of Squares

Recall the following formulas. They are all special cases of Binomial Theorem!

- $(a+b)^2 = a^2 + 2ab + b^2$ (square of a sum)
- $(a-b)^2 = a^2 - 2ab + b^2$ (square of a difference)
- $a^2 - b^2 = (a+b)(a-b)$ (difference of squares)
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ (cube of a sum)
- $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ (cube of a difference)

Binomial Probabilities

These numbers are also useful in calculating probabilities. Imagine that we have some event that happens with probability p ("success") and does not happen with probability $q = 1-p$ ("failure"). Then the probability of getting k successes in n trials is

$$P(k \text{ successes in } n \text{ trials }) = \binom{n}{k}p^k q^{n-k}, \text{ where}$$

- p — probability of success in one try;
- $q = 1-p$ — probability of failure in one try;
- n — number of trials;
- k — number of successes;
- $n-k$ — number of failures.

Example: You roll a dice 100 times. What is the probability of getting a 6 exactly 20 times?
Solution: Here we have: $n = 100$, $k = 20$, $p = 1/6$, $q = 5/6$. Then

$$P = \binom{100}{20} \cdot \left(\frac{1}{6}\right)^{20} \left(\frac{5}{6}\right)^{80}$$

Homework

1. Expand $(a+1)^7$

2. Expand $(x+2y)^4$

3. Expand $(2x-1)^5$

4. Expand $(2x+3y)^3$

5. What is the coefficient of the term x^9 in $(x^2-2x)^6$?

6. A test consists of 10 multiple choice questions with five choices for each question. As an experiment, you GUESS on each and every answer without even reading the questions. What is the probability of getting exactly 6 questions correct on this test?

Extra Problems (Optional)

1. Expand $\left(x + \dfrac{1}{x}\right)^3$

2. Use the Binomial Theorem to prove the identity:
$\binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n-1} + \binom{n}{n} = 2^n$

3. Compute $(a + b + c)^2$, $(a + b - c)^2$

4. Compute $(a^2 + 2ab + b^2)(a^2 - 2ab + b^2)$

5. In a certain city, 30% of the voters prefer candidate A over the others. If 10 voters are chosen randomly, what is the probability that 30% of them prefer candidate A?

6. In a heads or tails game, a try consists of tossing a coin three consecutive times. The "try" is considered a success if one gets strictly more heads than tails. What is the probability that one will succeed in the first two tries?

7. A box contains 1 black ball and 9 white balls. A second box also contains 10 balls, x of which are black and the others are white. Superman takes one ball of each box randomly. Batman puts the balls of both boxes together in a large box and then takes two balls randomly. What is the minimum value of x such that Batman is more likely to get two black balls than Superman?