MATH 7
ASSIGNMENT 10: TRIGONOMETRIC EQUATIONS AND INEQUALITIES
DEC 8, 2019

Tangent in the Trigonometric Circle

Just like with the sine and the cosine, the tangent can be defined through the trigonometric circle. Given a point (with coordinates $(\cos\alpha, \sin\alpha)$) corresponding to the angle α, one draws a line which passes through the origin and through this point, the tangent of α is the position along a line tangent to the circle through the point $(1, 0)$ of the intersection of these two lines (see figure 1). For angles $0 < \alpha < \pi/2$, this agrees with the previous. For other angles, we will use this procedure to define the tangent.

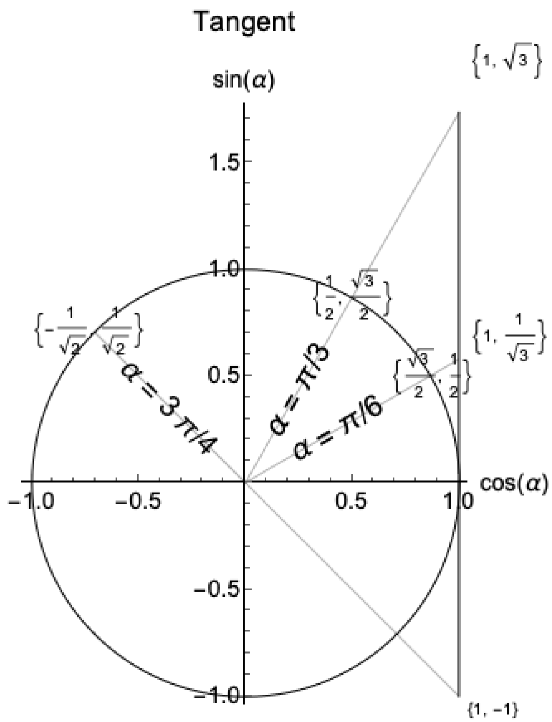

FIGURE 1. Meaning of $\tan\alpha$ in the trigonometric circle.

Compare this definition with the values we know from the table:

Trigonometric Functions							
Function	Notation	Definition	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sine	$\sin(\alpha)$	$\frac{\text{opposite side}}{\text{hypotenuse}}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cosine	$\cos(\alpha)$	$\frac{\text{adjacent side}}{\text{hypotenuse}}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tangent	$\tan(\alpha)$	$\frac{\text{opposite side}}{\text{adjacent side}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

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Trigonometric Equations and Inequalities

We can use our experience solving linear and quadratic equations and inequalities to solve analogous problems involving trigonometric functions.

There are important differences, however. For example, the periodicity of the trigonometric functions usually means that there are many solutions to an equation (an infinite number!). Let's look at a few examples:

Solving the equation $\sin x = \sin c$. Using the trigonometric circle (see figure 2), we see that, for a given c, the solution to the equation $\sin x = \sin c$ is

$$x = c + 2\pi * n$$

or

$$x = \pi - c + 2\pi * n,$$

where n can be any integer number (that's just because adding full turns doesn't change the sign!), which is why we have an infinite number of solutions.

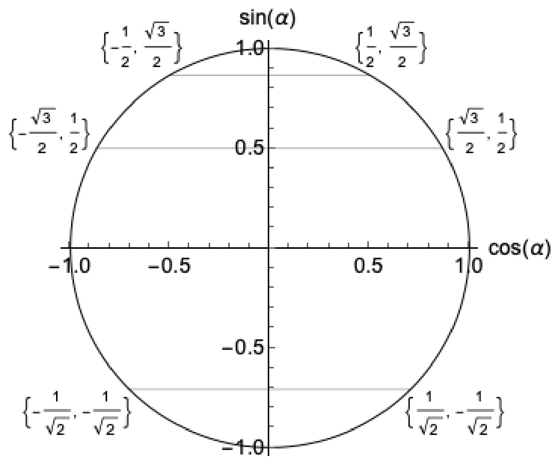

FIGURE 2. A few examples of the relation $\sin x = \sin(\pi - x)$.

Solving the equation $\cos x = \cos c$**.** By a similar inspection of the trigonometric circle (figure 3), we see that, for a given c, the solution to the equation $\cos x = \cos c$ is

$$x = c + 2\pi * n$$
or
$$x = -c + 2\pi * n,$$

where n can be any integer number.

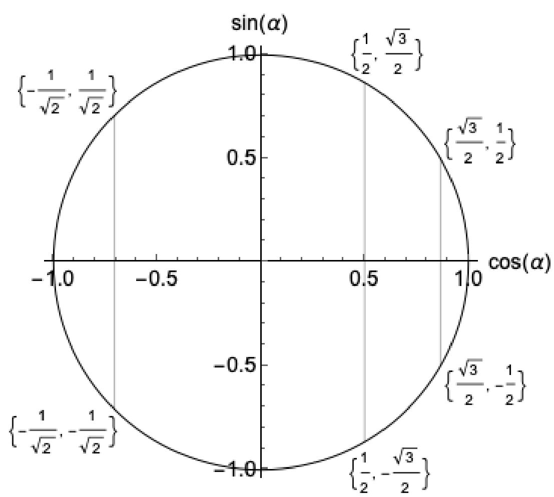

FIGURE 3. A few examples of the relation $\cos x = \cos(-x)$.

Solving the equation $\tan x = \tan c$**.** In this case (figure 4) we find that, for a given c, the solution to the equation $\tan x = \tan c$ is

$$x = c + 2\pi * n$$
or
$$x = c + \pi + 2\pi * n,$$

where n can be any integer number.

General Strategy. We see from these examples that, rather than trying to memorize all different cases, *one should always refer to the trigonometric circle*, from which the answer can be read off immediately. This is also true for trigonometric inequalities.

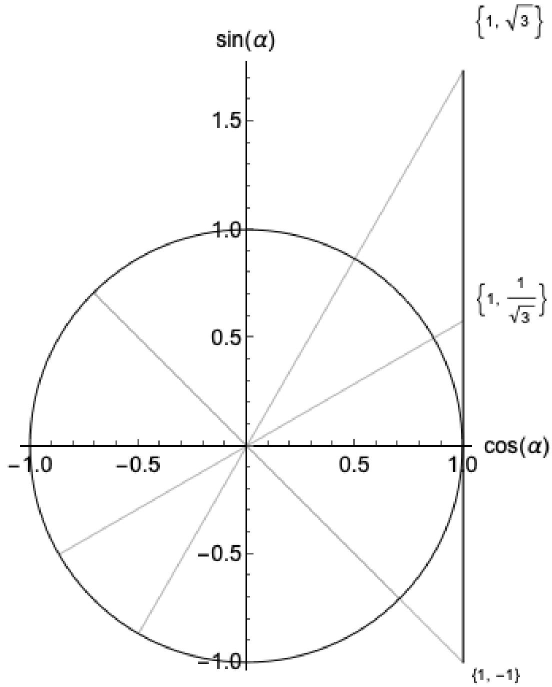

FIGURE 4. A few examples of the relation $\tan x = \tan(x + \pi)$.

Homework

1. Solve the following equations
   (a) $\sin x = \sin \frac{\pi}{5}$
   (b) $\sin x = 0$
   (c) $\cos x = \frac{\sqrt{2}}{2}$
   (d) $\cos(x + \frac{\pi}{6}) = 0$
   (e) $\sqrt{3} \tan x = 1$
   (f) $\tan x = -1$

2. Solve the following equations
   (a) $(\sin x)^2 - \sin x = 0$ [Hint: start by defining $y = \sin x$.]
   (b) $2(\sin x)^2 - 3 \sin x + 1 = 0$
   (c) $4 \cos x + \frac{3}{\cos x} = 8$
   (d) $(\sin x)^2 = (\cos x)^2$ [Hint: $\tan x = \sin x / \cos x$.]

3. Determine the internal angles of a triangle ABC knowing that the angles form an arithmetic sequence and that the sine of the sum of the two smallest angles is $\sqrt{3}/2$.

4. What are the real values of x for which
   (a) $\sin x > -\frac{\sqrt{2}}{2}$
   (b) $-\frac{1}{2} \leq \sin x < \frac{\sqrt{2}}{2}$
   (c) $2(\sin x)^2 < \sin x$
   (d) $\cos 2x \leq \frac{\sqrt{3}}{2}$
   (e) $(\tan x^2)^2 \leq 1$

5. From the previous assignment, you know how to sketch the graph of the sine and of the cosine functions in the interval from 0 to 4π. Now use the fact that $\tan x = \sin x / \cos x$ to sketch the graph of the tangent function. Does your sketch agree with the definition given above in terms of the trigonometric circle?