MATH 7
ASSIGNMENT 6: GEOMETRY OF THE PARABOLA
NOV 3, 2019

1. Focus and Directrix

A more geometrical definition of a Parabola is as the set of all points which are at equal distance from a given line **d**, called the **directrix** and a given point **P** (not in the directrix), called the **focus**.

Let us show that the graphs of quadratic functions, $y = ax^2 + bx + c$ satisfy this property. First we prove that we can rewrite this equation in the form $(x-h)^2 = 4p(y-k)$ for some coefficients h, p and k. We just have use our old trick of completing the square:

$$y = \quad ax^2 + bx + c = a\left[x^2 + \tfrac{bx}{a} + \tfrac{c}{a}\right] = a\left[\left(x + \tfrac{b}{2a}\right)^2 + \tfrac{c}{a}\right] = a\left[\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{D}{4a^2}\right]$$
$$\Rightarrow \tfrac{y}{a} + \tfrac{D}{4a^2} = \left(x + \tfrac{b}{2a}\right)^2 \Rightarrow \left(x + \tfrac{b}{2a}\right)^2 = \tfrac{1}{a}\left(y + \tfrac{D}{4a}\right),$$

which has the correct form $(x-h)^2 = 4p(y-k)$ with

(1) $$p = \frac{1}{4a}, \;\; k = -\frac{D}{4a} \quad \text{and} \quad h = -\frac{b}{2a},$$

with $D = b^2 - 4ac$, as usual.

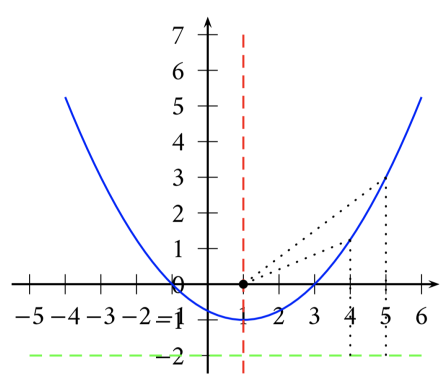
Now we note that any point (x, y) which satisfies $(x-h)^2 = 4p(y-k)$ is at the same distance from the point $P = (h, k+p)$ and from the line d given by $y = k - p$. Indeed, the (vertical) distance of a point (x, y) to the line d is

$$|y - (k-p)| = |y - k + p|,$$

and the distance of this point to $P = (h, k+p)$ is, by Pythagoras' Theorem,

$$\sqrt{(x-h)^2 + (y-(k+p))^2} = \sqrt{4p(y-k) + ((y-k)-p)^2} = \sqrt{4p(y-k) + (y-k)^2 - 2p(y-k) + p^2}$$
$$= \sqrt{(y-k)^2 + 2p(y-k) + p^2} = \sqrt{((y-k)+p)^2} = |y - k + p|,$$

so we see that the points (x, y) that satisfy $(x-h)^2 = 4p(y-k)$ or, equivalently, $y = ax^2 + bx + c$ (with the coefficients translated by equation (1)), lie at equal distance from the directrix $y = k - p$ and the focal point $P = (h, k+p)$.



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2. **Optical Properties**

Suppose one has a mirror in the shape of a Parabola. Then one can show that light rays incident in the parabola parallel to it's symmetry axis (the axis passing throught eh Focus in the direction normal to the directrix) are reflected passing through the Focus. This is why we use parabolic mirrors in telescopes!

Conversely, Light rays emitted (in all directions) from the Focus are reflected in the direction parallel to the symmetry axis. This is why parabolic reflectors are used in car headlights!

Homework

1. In each case, sketch the graph of the parabola, showing the roots, the Focus and the directrix.
   (a) $y = x^2 - 5x + 5$.
   (b) $y = x^2 - 5x - 14 = 0$.
   (c) $y = -x^2 + 2x + 2 = 0$.

2. Prove that for any point P on the parabola $y = \frac{x^2}{4} + 1$, the distance from P to the x-axis is equal to the distance from P to the point $(0, 2)$.

3. The perimeter of a rectangular room is 16m. Knowing that the room can fit at least 7 tables, each of which has a one squared meter area, what are the possible values for the length and width of the room?

4. A few university students devised a rocket prototype, but it failed. When launched, the rocket followed a trajectory $h = -d^2 + 101x - 100$, where h and d denote the height and d denotes the horizontal displacement on the ground from the launching point at each instant (both are measured in meters).
   (a) What is the shape of the trajectory?
   (b) Where did the rocket fall?
   (c) For which values of d was the rocket at a height bigger than 2000?

***5.** Consider the parabola $(x - h)^2 = 4p(y - k)$.
   (a) What is the point (x, y) of the parabola which is closest to the directrix? This is called the Vertex of the parabola. At what distance is it from the directrix? (This is why p is called the focal length of the parabola).
   (b) Consider now a line $y = p$, where p is some fixed number. For which values of p does the line: intersect the parabola twice? Intersect the parabola once? Doesn't intersect the parabola?
   (c) By the correspondence of the parabola $(x-h)^2 = 4p(y-k)$ with it's standard form $y = ax^2+bx+c$, where the coefficients are related by equation (1), relate the criteria from the previous part the one for the existence of roots of a quadratic equation in terms of the discriminant D (see Assignment 5).