QUADRATICS IN DISGUISE. VIETA FORMULAS

Definition *$a \in \mathbb{R}$ is a real **root** of a polynomial function $p(x)$ if $p(a) = 0$.*

Definition *$x - a$ is a **factor** of a polynomial function $p(x)$ if we can write $p(x) = q(x) \cdot (x - a)$ for some non-zero polynomial $q(x)$ with a degree $deg(q(x)) = deg(p(x)) - 1$.*

Theorem (The Factor Th.) $(x - a)$ is a factor of the polynomial (function) $p(x)$ if and only if $x = a$ is a root of $p(x)$.

From the Factor Theorem, if x_1 and x_2 are roots of the second degree polynomial function $f(x) = ax^2 + bx + c$, then $f(x) = a(x - x_1)(x - x_2) = a(x^2 - x_1x - x_2x + x_1x_2) = a(x^2 - (x_1 + x_2)x + x_1x_2)$. By definition, two polynomials are equal if and only if all their corresponding coefficients are equal. Thus:

Vieta formulas:

if $a = 1$, then

$$S = x_1 + x_2 = -b \text{ and } P = x_1x_2 = c$$

if $a \neq 1$, then

$$S = x_1 + x_2 = -\frac{b}{a} \text{ and } P = x_1x_2 = \frac{c}{a}$$

The Vieta formulas can be used to guess factors and find integer and rational roots.

Example 1. (Review) If α and β are the roots of the quadratic $x^2 - 4x + 9 = 0$, what are the values of

1. α, β
2. $\alpha + \beta$
2. $\alpha\beta$
3. $\alpha^2 + \beta^2$?

Sol:

1. Very often one cannot find α, β using Vieta relationships. When does it apply: if there is combination of integer factors of the product $P = 9$ having the sum $S = 4$. If this is not the case simply state it and try to complete to a square or directly use the discriminant formula.

2. From Vieta's formula, we have $\alpha + \beta = 4$.

3. From Vieta's formula, we have $\alpha\beta = 9$.

4. Vieta's formula does not give the value of $\alpha^2 + \beta^2$ directly. What we need to do is to write $\alpha^2 + \beta^2$ in terms of $\alpha + \beta$ and/or $\alpha\beta$, and we can then substitute these values in. We have

$$\begin{aligned} \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= 4^2 - 2 \times 9 \\ &= -2. \ \square \end{aligned}$$

Example 2.(Review) Which are the roots of the quadratic $x^2 + 5x + 6$?
Sol: Let p, q the roots . Using Vieta's formula we have

$$p + q = -5$$
$$pq = 6.$$

and we can easily see $-2 - 3 = -5$ and $(-2) \times (-3) = 6$. So the roots are -2 and -3
Example 3. Find a quadratic equation whose roots are 11 and -7. Is it unique? Why or why not?
Sol: Let the quadratic be $x^2 + bx + c$, where we wish to find b, c.Vieta's formula tells us that

$$b = -(11 + (-7)) = -4$$
$$c = 11 \times (-7) = -77$$

Therefore the desired quadratic equation is $x^2 - 4x - 77 = 0$. It is not unique. Think of multiplying it.

General formulas for the sum and product of roots

Vieta's formulas give a relationship between the roots of any polynomial and its coefficients, and yes they can be generalized for higher order polynomials. We are going to prove and use it for n=3, n=4
Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x^1 + a_0$ a polynomial of degree n, with roots $r_1, r_2, r_3, ..., r_n$
Then:

$s_1 = r_1 + r_2 + r_3 + ... + r_n = -\frac{a_{n-1}}{a_n}$
$s_2 = r_1r_2 + r_1r_3 + r_1r_4 + ... + r_{n-1}r_{n-2} = \frac{a_{n-2}}{a_n}$
......
$s_n = r_1r_2r_3r_4...r_n = (-1)^n \frac{a_0}{a_n}$

Homework

1. Factorize the following quadratics in disguise:

   (a) $x^8 + 2x^4 - 4$
   (b) $x^6 + 5x^3 + 6$
   (c) $x^4 + 2x^2 - 8$
   (d) $x + 7\sqrt{x} + 6$
   (e) $\frac{1}{x^2} - 7\frac{1}{x} + 10$
   (f) $\frac{1}{x^4} - 7\frac{1}{x^2} + 10$
   (g) $x^{\frac{2}{5}} + 3x^{\frac{1}{5}} + 2$
   (h) $x^{\frac{2}{3}} - 9x^{\frac{1}{3}} + 8$
   (i) $(x + 1)^2 - 7x - 7 + 12$
   (j) $(x + 5)^4 + 7(x + 5)^2 - 6$

2. Find the solutions of the following polynomial equations(use factoring):

   (a) $2x^3 - x^2 = 8x - 4$ (b) $x^3 + 5x^2 = 4x + 20$ (c) $8x^3 + 4x^2 = 18x + 9$ (d) $x^4 + 4x^3 + 4x^2 = -16x$

3. Let r and t be the roots of the quadratic equation $16x^2 - 5x + 1 = 0$. Find:

   (a) $r + t$ (b) $r^2 + t^2$ (c) $1/r + 1/t$ (d) $r^3 + t^3$ (e) $r^3 - t^3$

4. Solve in real numbers the system of equations : $x + y = 2$ and $xy = -2$.

5. What is the average of the values of x that satisfy the equation $x^2 + 432x + 169 = 0$?

6.* If a and b are the roots of the equation $x^2 - 5x + 6$, then what is the value of $(\sqrt{a} + \sqrt{b})(\sqrt{a} + \sqrt{b})$?

7.* Let x and y two distinct real numbers such that $x^2 + 3x + 1 = 0$ and $y^2 + 3y + 1 = 0$. Find $x + y$ and $\frac{x}{y} + \frac{y}{x}$.

8.* Let x_1, x_2 the solutions of the equation $3x^2 + 5x - 6 = 0$. Without solving the equation, make a quadratic equation with solutions $y_1 = x_1 + \frac{1}{x_2}$ and $y_2 = x_2 + \frac{1}{x_1}$

9.* Let $x^2 + 8x + k = 0$. Find k such that the sum of solutions of the given equation is ten times the positive difference its solutions.

10.* Prove the Vieta relations for n=3 and n=4.

[Optional Exercises for Mathematical Competitions]

11.* Let x_1, x_2, x_3 be the roots of equation $x^3 - x - 1 = 0$. Compute $\frac{2015+x_1}{2015-x_1} + \frac{2015+x_2}{2015-x_2} + \frac{2015+x_3}{2015-x_3}$

Hint: $\frac{2015+x}{2015-x} = \frac{4030}{2015-x} - 1$

12.* Let $g(x) = x^2 + bx + 4000$. One of the x-intercepts of $g(x)$ is four times another. What is b?

13.** Let $g(x) = x^3 + cx + 2500$. One of the x-intercepts of $g(x)$ is four times another. What is c?

14. Find the sum of the roots of $z^{20} - 19z + 2$.

15.* Find the sum of the 20th powers of the roots of $z^{20} - 19z + 2$.