CHAPTER 13

ROOTS OF POLYNOMIALS

Definition: a is a ***root of a polynomial*** $P(x)$ if $P(a) = 0$.
[!] *The x-intercept is that value of x where the graph of $y = P(x)$ touches the x axis $\Longrightarrow$ At x intercept, $y = P(x) = 0 \Longrightarrow$ x-intercept are roots.*

13.1 Polynomial division. Factor Theorem

Do you remember the long division in arithmetics ?

$$\text{dividend} = \text{quotient} \times \text{divisor} + \text{remainder}, \qquad \text{with } 0 \leq \text{remainder} < \text{divisor}$$

Let us try the same for polynomials divided by monomials.
Ex. 1:

$$f(x) = 2x^2 - 7x + 3,$$
$$g(x) = x - 3,$$

$$f(x) \div g(x)$$

$$\begin{array}{r|l} & 2x - 1 \\ \hline x - 3 & 2x^2 - 7x + 3 \\ & \underline{2x^2 - 6x} \\ & \quad -x + 3 \\ & \quad \underline{-x + 3} \\ & \qquad = \end{array}$$

Ex. 2:

$$f(x) = 2x^2 + 7x + 3,$$
$$g(x) = x - 2,$$

$$f(x) \div g(x)$$

$$\begin{array}{r|l} & 2x + 11 \\ \hline x - 2 & 2x^2 + 7x + 3 \\ & \underline{2x^2 - 4x} \\ & \quad 11x + 3 \\ & \quad \underline{11x - 22} \\ & \qquad +25 \end{array}$$

Independent classwork:
Ex. 3:

$$f(x) = x^2 - 3x - 10 \qquad\qquad g(x) = x - 5$$

So in general we have

The Factor Theorem: $x - a$ is a factor of a polynomial $P(x) \iff a$ is a root of $P(x)$.
Proof:

"$\Longrightarrow$" $(x - a)$ is a factor $\Longrightarrow P(x) = (x - a) \cdot (\dots) \Longrightarrow P(a) = 0 \cdot (\dots) \Longrightarrow a$ is a root.

"$\Longleftarrow$" a root of $P(x) \Longrightarrow P(a) = 0$, but $P(x) = Q(x) \cdot (x - a) + R$ and according to the remainder theorem, the remainder is a number $\Longrightarrow R = 0$.

13.2 Polynomial factoring

Ex. 4:

Use the Factor Theorem to prove that $(x + 1)$ is a factor of $x^5 + 1$.

Proof:

-1 is a root of $x^5 + 1$ since $(-1)^5 + 1 = 0 \Longrightarrow$ according to the Factor Theorem, $(x + 1)$ is a factor.

We can prove it using the division of polynomials:

$$\begin{array}{r|l} & x^4 - x^3 + x^2 - x + 1 \\ \hline x + 1 & x^5 + 0x^4 + 0x^3 + 0x^2 + 0x + 1 \\ & \underline{x^5 + x^4} \\ & \quad -x^4 \\ & \quad \underline{-x^4 - x^3} \\ & \qquad\quad x^3 \\ & \qquad\quad \underline{x^3 + x^2} \\ & \qquad\qquad -x^2 \\ & \qquad\qquad \underline{-x^2 - x} \\ & \qquad\qquad\qquad x \\ & \qquad\qquad\qquad \underline{x + 1} \\ & \qquad\qquad\qquad\ = \end{array}$$

Thus $x^5 + 1 = (x + 1)(x^4 - x^3 + x^2 - x + 1)$.

Independent classwork:
Ex. 5:

Construct a polynomial having as the only roots -1, 1 and 3.

13.3 Vieta's formulas (Viète's formulas) for quadratic equations

$y = ax^2 + bx + c$ is a polynomial function of second degree.

$ax^2 + bx + c = 0$ is a quadratic equation.

From the Factor Theorem, if x_1 and x_2 are roots of the second degree polynomial function $f(x) = ax^2 + bx + c$, then $f(x) = a(x - x_1)(x - x_2) = a(x^2 - x_1 x - x_2 x + x_1 x_2) = a(x^2 - (x_1 + x_2)x + x_1 x_2)$. Thus, if $a = 1$, then $x_1 + x_2 = -b$ and $x_1 x_2 = c$.

These formulas can be used to guess factors and find roots.

13.4 Problems

1. Factor $x^2 + 18x + 81$ using $(a + b)^2 = a^2 + 2ab + b^2$.
   We observe that it fits the pattern of a perfect square:
   $$\begin{cases} x_1 + x_2 &= -18 \\ x_1 \cdot x_2 &= 81 \end{cases} \quad \Longrightarrow \quad x_1 = -9, \quad x_2 = -9$$
   $\Longrightarrow (x + 9)^2 = x^2 + 18x + 81.$
2. Factor $x^2 + 10x + 21$ using Viète's formulas.
   $$\begin{cases} x_1 \cdot x_2 &= 21 \\ x_1 + x_2 &= -10 \end{cases} \quad \Longrightarrow \quad x_1 = -3, \quad x_2 = -7$$
   $\Longrightarrow (x + 3)(x + 7) = x^2 + 10x + 21,$
   $x(x + 3) + 7(x + 3) = x^2 + 3x + 7x + 21$ (split the middle)
3. Factor $x^2 + x - 12$ using Viète's formulas.
   $$\begin{cases} x_1 \cdot x_2 &= -12 \Longrightarrow x_1 = -3\,,\ x_2 = 4 \text{ or } x_1 = 3,\ x_2 = -4 \\ x_1 + x_2 &= -1 \end{cases} \Bigg| \quad \Longrightarrow \quad x_1 = 3, \quad x_2 = -4$$
   $\Longrightarrow (x - 3)(x + 4) = x^2 + x - 12,$
   $x^2 + x - 12 = x^2 + 4x - 3x - 12 = (x^2 + 4x) - (3x + 12) = x(x + 4) - 3(x + 4) = (x - 3)(x + 4)$

13.5 Completing the square

The "completing the square" method uses the formula

$$(x + y)^2 = x^2 + 2xy + y^2$$

For instance, we can rewrite:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 7 = (x + 3)^2 - 7$$

and then, $x^2 + 6x + 2 = 0$ if and only if $(x + 3)^2 = 7$, which gives $x + 3 = \sqrt{7}$ or $x + 3 = -\sqrt{7}$, thus the roots are $x_1 = -3 - \sqrt{7}$ and $x_2 = -3 + \sqrt{7}$.
So, more generally but supposing $a = 1$, we can write

$$x^2 + bx + c = x^2 + 2 \cdot \frac{b}{2} \cdot x + c = x^2 + 2 \cdot \frac{b}{2} \cdot x + \frac{b^2}{2^2} - \frac{b^2}{2^2} + c = \left(x + \frac{b}{2}\right)^2 - \frac{D}{4}, \text{ where } D = b^2 - 4c.$$

Now, solving $x^2 + bx + c = 0$ becomes equivalent to solving

$$\left(x + \frac{b}{2}\right)^2 = \frac{D}{4}$$

For the general case (a is not 1, nor zero), we can first divide everything by a, to equivalently solve

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \Longleftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}\,, \text{ and now } D = b^2 - 4ac$$

In order to have solutions, we need $D \geq 0$. Otherwise, if $D < 0$, there are no solutions. So, when $D \geq 0$,

$$x + \frac{b}{2a} = \pm\frac{\sqrt{D}}{2a} \Longleftrightarrow x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$$

13.6 Homework problems

1. Simply the following quotients using polynomial division:

   (a) $\dfrac{x^4 + 3x^3 - x^2 - x + 6}{x + 3}$

   (b) $\dfrac{2x^4 - 5x^3 + 2x^2 + 5x - 10}{x - 2}$

   (c) $\dfrac{x^3 - 2x^2 + 2x - 4}{x - 2}$

   (d) $\dfrac{x^3 - 1}{x - 1}$

2. Factor using Viète's formulas the following second degree polynomials

   (a) $x^2 + 9x + 14$

   (b) $x^2 - 6x - 7$

   (c) $8x^2 - 24x + 16$

   (d) $y^4 + y^2 - 12$

3. Think how should Viète's formulas look for

$$ax^3 + bx^2 + cx + d = 0$$