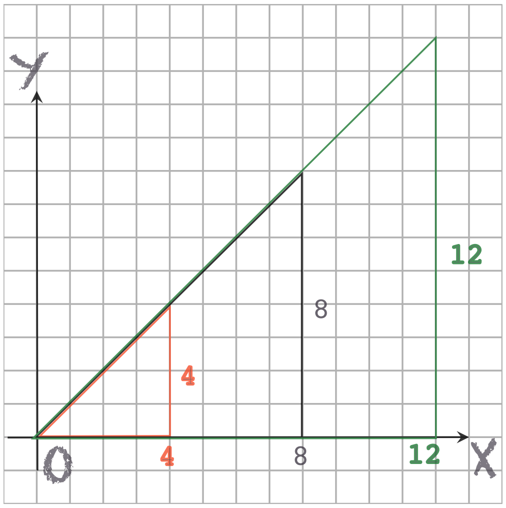
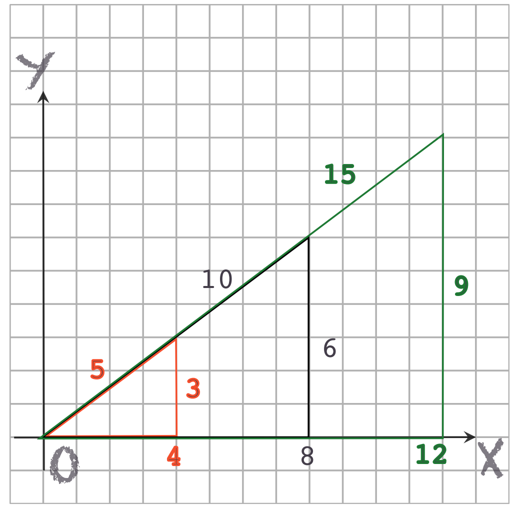


(a) Right Triangle 90,45,45



(b) Right Triangle 3,4,5

CHAPTER 5

BASIC TRIGONOMETRIC RATIOS

5.1 Warm-Up:

1. $1 + 3 + 5 + \cdots + (2k + 1) = ?$

2. In a set of five different numbers, which of the following operations can affect the median[1] of the numbers? Justify your answer. Operations : a) Increasing the smallest number only, b) Increasing the largest number only, c) Decreasing the largest number only, d) Decreasing the smallesr number only, e) Increasing each number by a constant, f) Multiplying each number with a constant

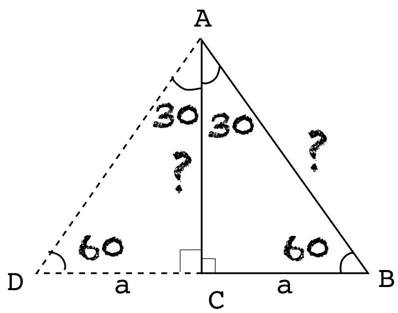
5.1.1 Measurements in the triangle

Trigonometry comes from "trigonon" which means in Greek "triangle" and "metron" which means "measure" and studies relationships involving lengths and angles of triangles. The trigonometric ratios are indeed special measurements of a right triangle. Let us follow the steps 3rd century BCE Greeks and take three nested triangles. They are all similar [2](i.e. have the same shape) with the same angles but they are different sizes. Indeed the values of the following fractions/ratios2 are the same although the lengths of the sides of the triangles are different.

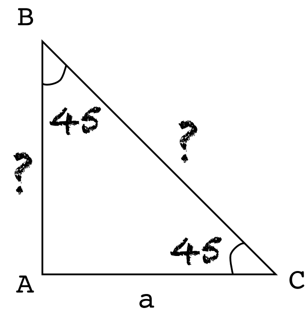
$$\frac{\text{opposite side}}{\text{hypothenuse}} = \frac{a}{c} = \frac{3}{5}, \frac{\text{adjacent side}}{\text{hypothenuse}} = \frac{b}{c} = \frac{4}{5}, \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b} = \frac{3}{4}$$

[1]Mean, median, and mode are three kinds of "averages". The mean is the number obtained after you add up all the numbers and then divide them by how many they are. The median is the middle value in the ordered list of numbers. The mode is the value that occurs most often. If no number is repeated, then there is no mode for the list.

[2]Euclid's "Elements" proves that if triangles are similar then the ratio of any chosen pair of sides stays the same, whatever the size of triangle. Most high-school textbooks do not prove this theorem, taking it as an additional postulate (often called the AAA Similarity Postulate). (Proof http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf Book VI, Proposition 4.)



(a) Right Triangle 90,60,30



(b) Right Triangle 90,45,45

The ratio of the opposite side of a right triangle to the hypotenuse is called the sine and denoted sin. The ratio of the adjacent side of a right triangle to the hypotenuse is called the cosine and denoted cos. Finally, the ratio of the opposite side to the adjacent side is called the tangent and denoted tan.

Connections:

$$tan(\alpha) = m_{BC}, \text{ the tangent equals the slope of the hypotenuse BC}$$

$$tan(\alpha) = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\text{opposite side}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{sin(\alpha)}{cos(\alpha)}$$

Trigonometric Functions							
Function	Notation	Definition	Acute Angle 3,4,5	0	30	45	60
Sine	$sin(\alpha)$	$\frac{\text{opposite side}}{\text{hypotenuse}}$		0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
Cosine	$cos(\alpha)$	$\frac{\text{adjacent side}}{\text{hypotenuse}}$		$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
Tangent	$tan(\alpha)$	$\frac{\text{opposite side}}{\text{adjacent side}}$		0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

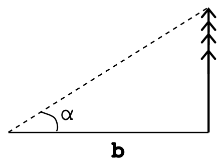
Mnemonics: For the sine the numerator is $\sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}$

The calculations for the 45-45-90 and for the 30-60-90 right triangles

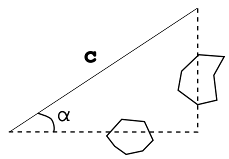
The 30-60-90 triangle and its mirror reflection create together a 60-60-60 triangle of side equal to the hypothenuse.

What about other angles? It might be tempting to use for the expression sin(a + b) the sum sin(a) + sin(b), but this is completely wrong. **Tables of values have been calculated for the sin, cos and tan of many angles.**

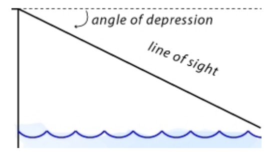
Trigonometry is largely used in land surveying. When an observer looks at an object that is higher than the (eye of) the observer, the angle between the line of sight and the horizontal is called the angle of elevation. On the other hand, when the object is lower than the observer, the angle between the horizontal and the line of sight is called the angle of depression. These angles are always measured from the horizontal.



(a) Exercise 1: Using shadows to measure



(b) Exercise 2: Measure lengths despite obstacles



(c) Exercise 5: Angle of depression

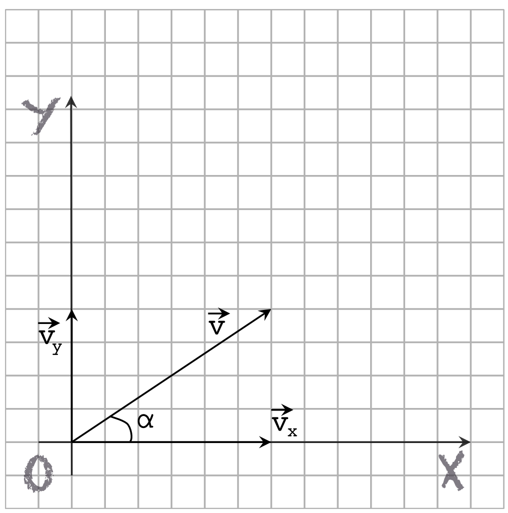
Land Surveying Problems

5.1.2 Horizontal and vertical components of vectors : $\vec{v} = \vec{v_x} + \vec{v_y}$

If we know a vector $\vec{v}$ and the angle it forms with the positive x-axis, OX then we can find out its vertical and horizontal components using the sine and cosine in scalar equations.:

$$\text{the x-component of } \vec{v} : v_x = |v| \cdot cos(\alpha) \text{ , the y-component of } \vec{v} : v_y = |v| \cdot sin(\alpha)$$

If we know the components of a vector $\vec{v}$, we can determine the magnitude or length of the vector $|v| = \sqrt{{v_x}^2 + {v_y}^2}$. Considering the components as vectors on a line, the law for adding vectors says $\vec{v} = \vec{v_x} + \vec{v_y}$.



5.2 Problems

1. Find the values of sine, cosine, tangent of the angle C of the right triangle ΔABC with ($\angle A = 90$) if
   (a) AB=5, AC=8
   (b) AB=60, BC=100

2. (Using shadows to measure things) A tree casts a 60 m shadow when the angle of elevation of the sun is 30. What is the height of the tree? (see Figure Exercise 1:)(i.e. the elevation angle is the angle at which you look up to the top of the tree from the end of the shadow on the ground)

3. (Measuring across obstacles) A land surveyor needs to measure the lengths of two rivers with a common source. The rivers have inaccessible parts due to fast rapids, but the angle between the rivers is known to be of 90 degrees. What are the rivers' lengths, if later the distance between the two rivers is c= 25 and a base angle is $\alpha = 30$? (see Figure Exercise 2)

4. (Elevation angle) A hot air balloon is rising at a rate of 6 ft/sec, while a wind is blowing E-W at a rate of $2\sqrt{3}$ft/sec. Find the speed at which the balloon is traveling, draw its velocity vector, and find its angle of elevation.

5. (Depression angle) The Montauk lighthouse is 110.5 ft high and it is situated on a natural foundation of 14 ft. Its keeper sights from the top of the lighthouse a sailboat. The angle of depression of the sighting is 30 degrees. How far is the sailboat ? (see Figure Exercise 5)

6. In a science camp a group of students are building rockets that are afterwards rockets at different angles. An observer situate at 1 m away from the launching point measures the height reached by the rocket. What will be the height measured for a rocket launched at an angle of 30 degrees? What will be the height measured for a rocket launched at an angle of 60 degrees? What happens when launching angle gets closer to 90 degrees?

7. A ball is kicked into the air at an angle of 30 degrees to the horizontal with an initial resultant velocity of 25 m/s. Find both the vertical and horizontal components of the velocity vector and the magnitude of the resultant velocity vector.

8. A helicopter is travelling at a speed of 100m/s towards a destination that is located 30 degrees north of east. What are its velocity components due east and due north.

9. A plane is taking off from the airport strip with a speed of 250 m/s at an angle of 60 degrees. How much altitude will it gain in 1 second?

10. One day you stroll down to the river and take a walk along the river bank. At one point in time you notice a rock directly across from you. After walking 100 feet downstream, you have to turn an angle of 30 with the river to be looking directly at the rock. How wide is the river?

11.* Consider a parallelogram ABCD with AB = 1, AD = 3, A = 40 deg. Find the lengths of diagonals in this parallelogram.

    [Hint: introduce a coordinate system so that $\overrightarrow{AD}$ goes along the x-axis. For the diagonal AC write the vector $\overrightarrow{AC}$ as a sum of two vectors, decompose $\overrightarrow{AB} = \overrightarrow{BC}$ into horizontal and vertical components]

12.* Prove that the area of a triangle ΔABC can be computed using the formula $Area_{\Delta ABC} = \frac{AB \cdot AC \cdot sinA}{2}$
    [Hint: what is the altitude from vertex B?]

Table of sin (angle)

Angle	sin (a)	Angle	sin (a)	Angle	sin (a)	Angle	sin (a)
0.0	0.0	25.0	.4226	46.0	.7193	71.0	.9455
1.0	.0174	26.0	.4384	47.0	.7314	72.0	.9511
2.0	.0349	27.0	.4540	48.0	.7431	73.0	.9563
3.0	.0523	28.0	.4695	49.0	.7547	74.0	.9613
4.0	.0698	29.0	.4848	50.0	.7660	75.0	.9659
5.0	.0872	30.0	.5000	51.0	.7772	76.0	.9703
6.0	.1045	31.0	.5150	52.0	.7880	77.0	.9744
7.0	.1219	32.0	.5299	53.0	.7986	78.0	.9781
8.0	.1392	33.0	.5446	54.0	.8090	79.0	.9816
9.0	.1564	34.0	.5592	55.0	.8191	80.0	.9848
10.0	.1736	35.0	.5736	56.0	.8290	81.0	.9877
11.0	.1908	36.0	.5878	57.0	.8387	82.0	.9903
12.0	.2079	37.0	.6018	58.0	.8480	83.0	.9926
13.0	.2249	38.0	.6157	59.0	.8571	84.0	.9945
14.0	.2419	39.0	.6293	60.0	.8660	85.0	.9962
15.0	.2588	40.0	.6428	61.0	.8746	86.0	.9976
16.0	.2756	41.0	.6561	62.0	.8829	87.0	.9986
17.0	.2924	42.0	.6691	63.0	.8910	88.0	.9994
18.0	.3090	43.0	.6820	64.0	.8988	89.0	.9998
19.0	.3256	44.0	.6947	65.0	.9063	90.0	1.00
20.0	.3420	45.0	.7071	66.0	.9135		
21.0	.3584			67.0	.9205		
22.0	.3746			68.0	.9272		
23.0	.3907			69.0	.9336		
24.0	.4067			70.0	.9397		

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Table of cos(angle)

Angle	cos(a)	Angle	cos(a)	Angle	cos(a)	Angle	cos(a)
0.0	1.00	25.0	.9063	46.0	.6947	71.0	.3256
1.0	.9998	26.0	.8988	47.0	.6820	72.0	.3090
2.0	.9994	27.0	.8910	48.0	.6691	73.0	.2924
3.0	.9986	28.0	.8829	49.0	.6561	74.0	.2756
4.0	.9976	29.0	.8746	50.0	.6428	75.0	.2588
5.0	.9962	30.0	.8660	51.0	.6293	76.0	.2419
6.0	.9945	31.0	.8571	52.0	.6157	77.0	.2249
7.0	.9926	32.0	.8480	53.0	.6018	78.0	.2079
8.0	.9903	33.0	.8387	54.0	.5878	79.0	.1908
9.0	.9877	34.0	.8290	55.0	.5736	80.0	.1736
10.0	.9848	35.0	.8191	56.0	.5592	81.0	.1564
11.0	.9816	36.0	.8090	57.0	.5446	82.0	.1392
12.0	.9781	37.0	.7986	58.0	.5299	83.0	.1219
13.0	.9744	38.0	.7880	59.0	.5150	84.0	.1045
14.0	.9703	39.0	.7772	60.0	.5000	85.0	.0872
15.0	.9659	40.0	.7660	61.0	.4848	86.0	.0698
16.0	.9613	41.0	.7547	62.0	.4695	87.0	.0523
17.0	.9563	42.0	.7431	63.0	.4540	88.0	.0349
18.0	.9511	43.0	.7314	64.0	.4384	89.0	.0174
19.0	.9455	44.0	.7193	65.0	.4226	90.0	0.0
20.0	.9397	45.0	.7071	66.0	.4067		
21.0	.9336			67.0	.3907		
22.0	.9272			68.0	.3746		
23.0	.9205			69.0	.3584		
24.0	.9135			70.0	.3420		

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Table of tan(angle)

Angle	tan(a)	Angle	tan(a)	Angle	tan(a)	Angle	tan(a)
0.0	0.00	25.0	.4663	46.0	1.0355	71.0	2.9042
1.0	.0175	26.0	.4877	47.0	1.0724	72.0	3.0777
2.0	.0349	27.0	.5095	48.0	1.1106	73.0	3.2709
3.0	.0524	28.0	.5317	49.0	1.1504	74.0	3.4874
4.0	.0699	29.0	.5543	50.0	1.1918	75.0	3.7321
5.0	.0875	30.0	.5773	51.0	1.2349	76.0	4.0108
6.0	.1051	31.0	.6009	52.0	1.2799	77.0	4.3315
7.0	.1228	32.0	.6249	53.0	1.3270	78.0	4.7046
8.0	.1405	33.0	.6494	54.0	1.3764	79.0	5.1446
9.0	.1584	34.0	.6745	55.0	1.4281	80.0	5.6713
10.0	.1763	35.0	.7002	56.0	1.4826	81.0	6.3138
11.0	.1944	36.0	.7265	57.0	1.5399	82.0	7.1154
12.0	.2126	37.0	.7535	58.0	1.6003	83.0	8.1443
13.0	.2309	38.0	.7813	59.0	1.6643	84.0	9.5144
14.0	.2493	39.0	.8098	60.0	1.7321	85.0	11.430
15.0	.2679	40.0	.8391	61.0	1.8040	86.0	14.301
16.0	.2867	41.0	.8693	62.0	1.8907	87.0	19.081
17.0	.3057	42.0	.9004	63.0	1.9626	88.0	28.636
18.0	.3249	43.0	.9325	64.0	2.0503	89.0	57.290
19.0	.3443	44.0	.9657	65.0	2.1445	90.0	infinite
20.0	.3640	45.0	1.000	66.0	2.2460		
21.0	.3839			67.0	2.3559		
22.0	.4040			68.0	2.4751		
23.0	.4245			69.0	2.6051		
24.0	.4452			70.0	2.7475		

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