CHAPTER 5

VECTORS

5.1 Definition of a vector

Vectors are objects originating from physics, where many quantities (velocities, forces, etc) are naturally described by their size (i.e. magnitude) as well as by their direction. However, when only the size is of interest as for length, area, volume, mass and time, we use a real number with appropriate units.

Definition *A vector is an object that has both a magnitude (size) and a direction. Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction. The direction of the vector $\overrightarrow{AB}$ is from its tail, initial point A, to its head, terminal point B.*

In the Cartesian coordinate plane XOY, any plane vector can be given in a **component form** (v_x, v_y) described by two numbers, its x-coordinate and its y-coordinate. For a vector $\overrightarrow{AB}$, with $A = (x_1, y_1)$, and $B = (x_2, y_2)$

$$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1), \quad \text{magnitude of } \overrightarrow{AB} = distance(AB) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Standard Notation:

for the vectors: lower case letters with an arrow overline $\vec{v}$, or with an underline $\underline{v}$ or bold x, v (in books).

for their magnitude, norm, modulus or length of the vector: $v \equiv |\vec{v}|$.

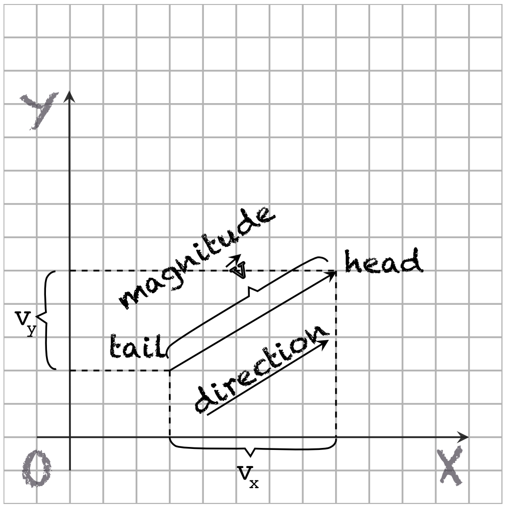
Example: For A(3,6) and B(6,2) we have

$$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1) = (6 - 3, 2 - 6) = (3, 4), \;\; AB = |\overrightarrow{AB}| = 5$$

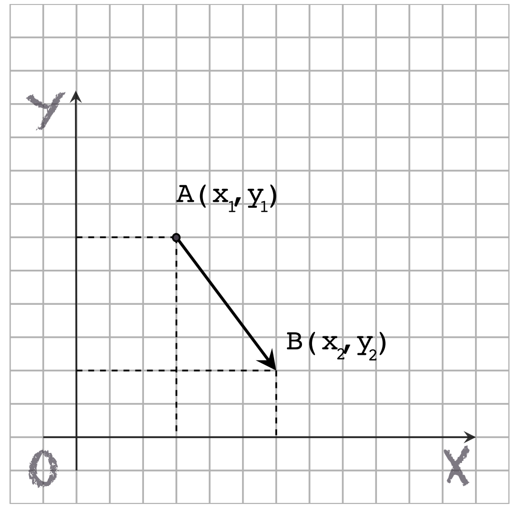
In physics it is natural that **the location of the vector** (the point at which a specific vector force is acting, or where a moving object with a given vector velocity is located) **is not part of the vector itself**. Thus in mathematics we also consider **two vectors to be the same if they have the same length and direction**. Graphically, two vectors which are the same form two opposite sides of a parallelogram. Thus, we can write any vector $\vec{v}$ as a vector with tail at a given point A. For its head we can use $A + \vec{v}$.

The vector $O + \vec{v}$, with tail at $O(0,0)$ is said to be in **standard position**. In this case the vector is completely determined by the position of its head, giving a correspondence between vectors and points in the plane.

Definition *A* ***position vector*** *of a point A is the vector with the origin O(0,0) as the tail, initial point, and A as the head, terminal point.*



(a) Vector



(b) Vector with tail A and head B

5.2 Operations with Vectors

- Addition beween two vectors by the **Triangle law**: $(\vec{v}, \vec{w}) \longmapsto \vec{v} + \vec{w}$ =third missing side of the triangle formed when the two vectors $\vec{v}$ and $\vec{w}$ are placed head to tail, oriented from the 1-st one head to the 2-nd one tail

  By constructing the parallelogram of same vectors, the defined sum of vectors is exactly the diagonal of the parallelogram formed when the two vectors $\vec{v}$ and $\vec{w}$ are placed at the same point. This property is known as the **Parallelogram rule** and graphically shows that the sum of two vectors is commutative, that is, the order of adding them is not important. In a plane coordinate system we have $\vec{v} + \vec{w} = (v_x + w_x, v_y + w_y)$.

- Multiplication by a real number In a plane coordinate system $(n, \vec{v} = (v_x, v_y)) \longmapsto n\vec{v} = (nv_x, nv_y)$.

  In the multiplication of a vector by a positive real number, the number acts as a scale factor giving a new vector with the same direction but with a magnitude which is multiplied by the positive real number. Thus, the real number is also called a scalar quantity. A negative scalar would yield a vector in the opposite direction as the original vector.

- There is no obvious elementary way of multiplying two vectors

5.3 Properties of Vector Operations

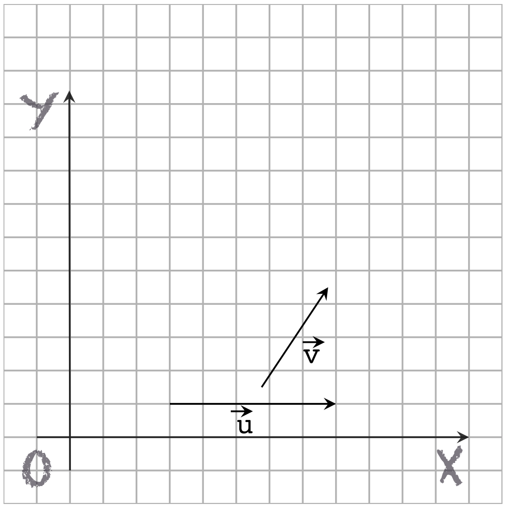
V1) **Commutativity** of addition : $\vec{v} + \vec{w} = \vec{w} + \vec{v}$, for all vectors $\vec{v}$, $\vec{w}$

V2) **Associativity**

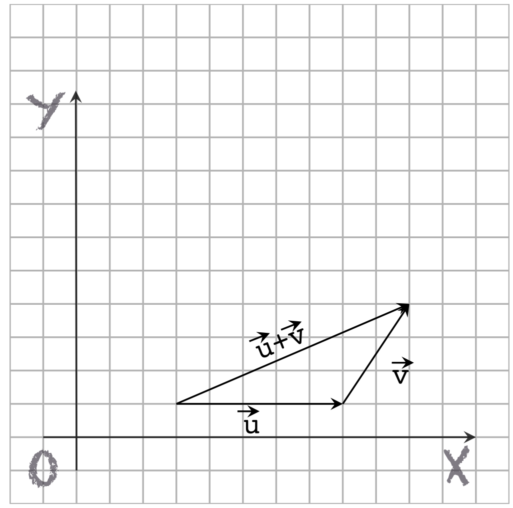
  (a) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$, for all vectors $\vec{u}$, $\vec{v}$, $\vec{w}$.
  Graphically, in a coordinate system it is easy to see that the vector addition is associative, that is, it does not matter whether we add the first pair first, or the second pair first.

  (b) $n(m\vec{v}) = (nm)\vec{v}$, for all $n, m \in \mathbb{R}$, for any vector $\vec{v}$

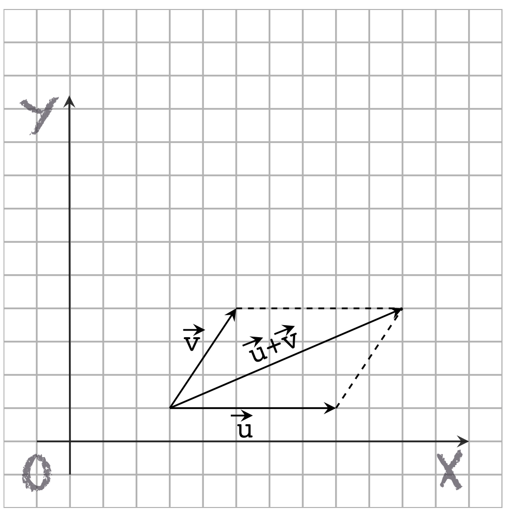
V3) Existence of a **neutral element** for addition:
there exists $\vec{0} = (0, 0)$, such that $\vec{v} + \vec{0} = \vec{v}$, for any vector $\vec{v}$



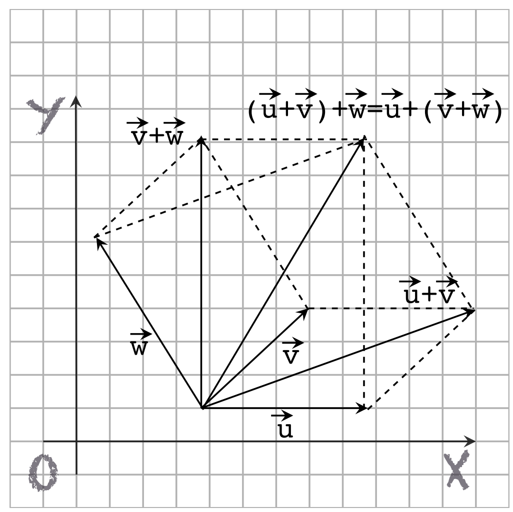
(a) Initial vectors



(b) Vector addition



(a) Parallelogram rule



(b) Associativity

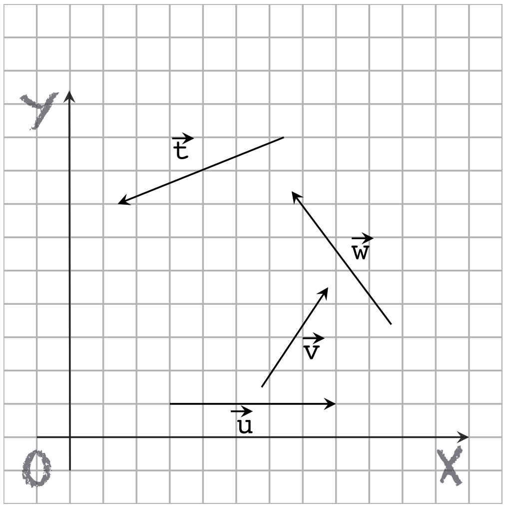
V4) Existence of **opposite vectors**:
for any $\vec{v}$ there exists $\vec{w}$ such that $\vec{v} + \vec{w} = \vec{0}$

V5) **Normalisation** $1 \cdot \vec{v} = \vec{v}$, for any vector $\vec{v}$

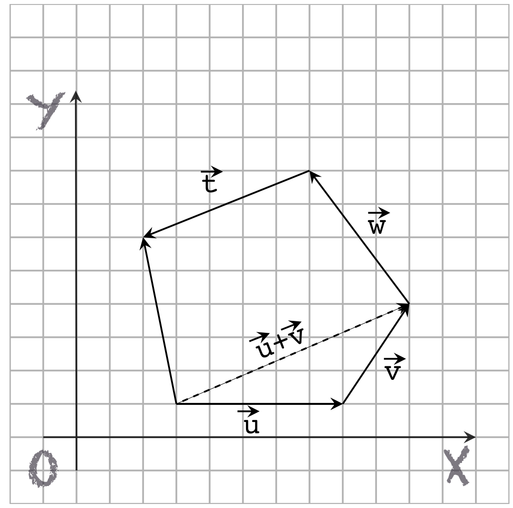
V6) **Distributivity**:

(a) $n(\vec{v} + \vec{w}) = n\vec{v} + n\vec{w})$, for all $n \in \mathbb{R}$, for all vectors $\vec{v}$, $\vec{w}$
(b) $(n + m)\vec{v} = n\vec{v} + m\vec{v}$, for all $n, m \in \mathbb{R}$, for any vector $\vec{v}$.

Adding more vectors? It is easy to prove the **Polygon rule**: If the n vectors placed head to tail form a convex polygon, their sum is given by the polygon's last missing side.



(a) Parallelogram rule



(b) Associativity

On a final note: the properties of vectors model the properties of some physical quantities. However, it is important to know that **not all physicall quantities with direction and magnitude are vectors.** Sometimes the movement described by those physical quantities is too complicated to support such simple addition, or multiplication rules. For example, angular displacement which describes the rotation of a rigid body over fixed axes in degrees is a quantity that has both direction and magnitude, but it is not a vector.

5.4 Problems

1. Consider the vector $\overrightarrow{AB} = (2,3)$ having as tail (i.e. initial point) the point $A = (1,1)$. Find its head (i.e. final point). Find the coordinates of a point P so that the position vector OP is the standard vector for AB.

2. Let $A(3,6)$, $B(5,2)$. Find the coordinates of the vector $\vec{v} = \overrightarrow{AB}$ and coordinates of the points $A + 2\vec{v}$, $A + \frac{1}{2}\vec{v}$, $A - \vec{v}$. What relationship seems to be between all these points ? Prove it and explain it. Repeat for points $A(x_1, y_1)$, $B(x_2, y_2)$.

3. If we translate the origin $O(0,0)$ to $(2,3)$, are the following statements true or false ? Explain.
   (a) The magnitude of a vector will be different in the new coordinate system.
   (b) The direction of a vector will be different in the new coordinate system.
   (c) The components of a vector will be different in the new coordinate system.

4. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $O(0,0)$. Show that the midpoint M of the segment AB has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ and that $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$. [Hint: point M is $A + \frac{1}{2} \cdot \vec{v}$, with $\vec{v} = \overrightarrow{AB}$.]

5. Consider a parallelogram ABCD with vertices $A(0,0)$, $B(3,6)$, $D(5,2)$. Find the coordinates of:
   (a) the vertex C and the midpoints of the segments BD and AC
   (b)* What geometrical fact did you prove for the parallelogram ABCD ? Now prove it in the case of arbitrary coordinates: $B(x_1, y_1)$, $D(x_2, y_2)$.

6.* Let $O(0,0)$, $A = (x_1, y_1)$, $B = (x_2, y_2)$ and a point R such that $n \cdot AR = m \cdot RB$. Find the coordinates of R and show that $\overrightarrow{OR} = \frac{1}{m+n}(n \cdot \overrightarrow{OA} + m \cdot \overrightarrow{OB})$.

7. Consider a ΔABC with $A(2,1)$, $B(3,8)$, $C(7,0)$.
   (a) Find the coordinates of the midpoints M, N, P of the segments BC, AC and respectively AB
   (b) Find the coordinates of the point on the median AM which divides AM in proportion 2:1. Repeat the same for the two other medians.
   (c) Prove that the three medians of the ΔABC intersect at a single point.

8.* Consider a ΔABC and take a point M as the midpoint of the side BC of the triangle.
   (a) Prove that the vector $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$
   (b) Prove that the medians AM, BN and CP intersect each other at a point G which divides AM in proportion 2:1.
   (c) Prove that $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$, where $O(0,0)$.
   (d) Take the other two midpoints N, P. Prove that the sum of vectors $\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CP} = \vec{0}$. Why do you think this is happening?

9.* Consider a convex closed n-sided polygon $A_1 A_2 \ldots A_{n-1} A_n$. Then the sum of the contour vectors $\overrightarrow{A_k A_{k+1}}$ is equal to zero.

10.* Prove that the sum of the squares of the sides of a parallelogram ABCD is equal to the sum of the squares of the diagonals AC respectively BD.
   [Hint1 : A parallelogram is a quadruple of vectors of the form $(\vec{a}, \vec{b}, \vec{a}, \vec{b})$)]
   [Hint2 : Choose a suitable coordinate system with A as origin, B on the positive X-axis]