MATH 6: HOMEWORK 21

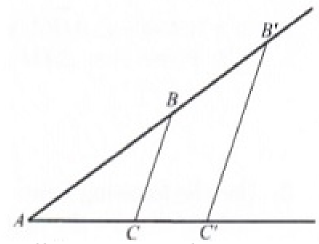
MAR 8, 2020

1. Similar triangles

We say that triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar with coefficient k if $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $\angle C = \angle C'$ and

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = k.$$

We will use notation $\triangle ABC \sim \triangle A'B'C'$.



Theorem 19. *Consider a triangle $\triangle ABC$ and let $B' \in \overrightarrow{AB}$, $C' \in \overrightarrow{AC}$ be such that lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel. Then $\triangle ABC \sim \triangle A'B'C'$.*

The proof of this theorem is actually quite hard. For this reason, we will not give it here; in most high school geometry courses, it is taken as an axiom.

Theorem 20. *For any triangle $\triangle ABC$ and a real number $k > 0$, there exists a triangle $\triangle A'B'C'$ similar to $\triangle ABC$ with coefficient k.*

Theorem 21 (Similarity via AA)**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that $m\angle A = m\angle A'$, $m\angle B = m\angle B'$. Then these triangles are similar.*

Proof. Let $k = \frac{A'B'}{AB}$. Construct a triangle $\triangle A''B''C''$ which is similar to $\triangle ABC$ with coefficient k. Then $A'B' = A''B''$, and $m\angle A = m\angle A' = m\angle A''$, $m\angle B = m\angle B' = m\angle B''$. Thus, by ASA, $\triangle A'B'C' \cong \triangle A''B''C''$. $\square$

Theorem 22 (Similarity via SAS)**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that $\angle A = \angle A'$, $\frac{A'B'}{AB} = \frac{A'C'}{AC}$. Then these triangles are similar.*

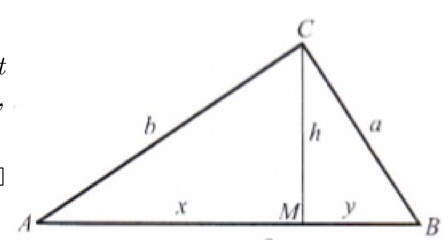
Theorem 23 (Similarity via SSS)**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that*

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$$

Then these triangles are similar.

One of the most important applications of the theory of similar triangles is to the study of right triangles and Pythagorean theorem (which we will discuss next time)

A right triangle is a triangle in which one of the angles is a right angle. A hypotenuse is the side opposing the right angle; two other sides are called legs.



Theorem 24. *Let $\triangle ABC$ be a right triangle, with $\angle C$ being the right angle. Let CM be the altitude of angle C. Then triangles $\triangle ABC$, $\triangle ACM$, $\triangle CBM$ are all similar.*

Proof. It immediately follows from AA similarity rule. $\square$

This theorem immediately implies a number of important relations between various lengths in these triangles. We will give one of them. Denote for brevity $a = BC$, $b = AC$, $c = AB$, $x = AM$, $y = MB$, $h = CM$. Then we have $x : h = b : a$, $y : h = a : b$, so

$$\frac{x}{h} \times \frac{y}{h} = 1$$

or $xy = h^2$.

Homework

1. Problems 25, 26, 32 on pages pages 230-231 in the textbook.

2. Show that two equilateral triangles are similar.

3. Show that two right isosceles triangles are similar.

4. Triangles ABC and $A'B'C'$ are similar. The angle bisectors of angles A and A' intersect BC and $B'C'$ at D and D'. Show that $\triangle ABD \sim \triangle A'B'D'$.

5. Triangles ABC and $A'B'C'$ are similar. Draw the medians AM and $A'M'$, where M is the midpoint of BC and M' is the midpoint of $B'C'$. Show that $\triangle ABD \sim \triangle A'B'D'$.

6. Same problem but with altitudes.

7. In a right triangle ABC, let AD be the altitude to the hypotenuse. Show that $AB^2 = BD \times BC$ and $AD^2 = BD \times DC$.

8. Let $ABCD$ be a trapezoid with bases $AD = 9$, $BC = 6$, such that the height (distance between the bases) is equal to 5. Let O be the intersection point of lines AB, CD.
(a) Show that triangles $\triangle OBC$, $\triangle OAD$ are similar and find the coefficient.
(b) Find the distance from O to AD (i.e., length of the perpendicular).

9. In a triangle $\triangle ABC$, let D be midpoint of side BC, E – midpoint of side AC, F – midpoint of side AB. Prove that $\triangle DEF$ is similar to triangle $\triangle ABC$ with coefficient 1/2.

10. Use the folowing figure to prove that an angle bisector in a triangle $\triangle ABC$ divides the opposite side in the same proportion as the two adjoining sides: $\frac{x}{y} = \frac{BA}{BC}$.