EUCLIDEAN GEOMETRY 6: PYTHAGOREAN THEOREM

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Pythagorean Theorem

In a right triangle with legs a and b, and hypotenuse c, the square of the hypotenuse is the sum of squares of each leg. $c^2 = a^2 + b^2$. The converse is also true, if the three sides of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle. This is indeed one of geometry's most famous theorems, and it is fundamentally important for many applications that involve measuring distances, for example in coordinate systems. There are dozens (if not hundreds) of proofs of the Pythagorean theorem; we will discuss applications of the theorem, and some problems related to its proofs will be left to you as homework.

If a collection of three numbers would fit as lengths of a right triangle, then the numbers are said to be a *Pythagorean Triple*. For example, (3,4,5) make a right triangle because $3^2 + 4^2 = 5^2$. Some other Pythagorean triples are: (5,12,13), (7,24,25), (8,15,17), (9,40,41), (11,60,61), (20,21,29).

Here is a method to generate Pythagorean triples: choose two positive integers a and b, $(a > b)$; then $(a^2 - b^2,\ 2ab,\ a^2 + b^2)$ is a Pythagorean triple. Can you explain why this method works?

The 30-60-90 Triangle

In a right triangle, if one of the angles is given as 30° or 60° then this triangle is called 30-60-90 triangle and you know the ratio of the sides. If the smaller leg is a then the hypotenuse is $2a$ and using Pythagorean theorem one can find the altitude to be $a\sqrt{3}$.

The 45-45-90 Triangle

Given that an angle of a right triangle is 45°, you can compute the other angle and it will also be 45°. This triangle is half a square, when the square is folded along its diagonal.

Distance between Two Points

The distance between points (x_1, y_1) and (x_2, y_2) in the xy-plane can be calculated using the Pythagorean theorem by forming an appropriate right triangle. Let the distance be d; then $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, thus $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. To prove this, draw a right triangle with legs equal to $(x_2 - x_1)$ and $(y_2 - y_1)$ in length, and the line segment from (x_1, y_1) to (x_2, y_2) as hypotenuse. Can you figure out how?

Midpoint of a Line Segment

The midpoint of a segment with endpoints (x_1, y_1) and (x_2, y_2) is the point with coordinates $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$.

Homework

1. (a) Find the distance between (5,-4) and (0,-1) on the coordinate plane. What is the midpoint?

(b) Find the distance between (5,-4) and (-3,2) on the coordinate plane. What is the midpoint?

2. In a trapezoid ABCD with bases AD and BC, $\angle A = 90°$, and $\angle D = 45°$. It is also known that $AB = 10$ cm, and $AD = 3BC$. Find the area of the trapezoid.

3. A circle is inscribed in an equilateral triangle. What fraction of the area of the triangle is inside the circular region? Express your answer as a common fraction in simplest radical form in terms of π.

4. In a right triangle ABC, BC is the hypotenuse. Draw AD perpendicular to BC, where D is on BC. The length of BC=13, and AB=5. What is the length of AD?

5. What is the area of a regular hexagon whose side is 2cm?

6. Prove that, in a right triangle where one of the acute angles is 60°, the leg next to the 60° angle is half the length of the hypotenuse.

7. What is the sum of the first 100 positive odd integers?

***8.** Given 2 concentric circles, chord AB is 8cm long and tangent to the smaller of two concentric circles. A and B are points on the larger circle. What is the area between the 2 circles?

***9.** Let $\triangle ABC$ be a right triangle with hypotenuse BC, and let AD be the altitude from A to the hypotenuse.