MATH 6: EUCLIDEAN GEOMETRY 5

MAR 8, 2020

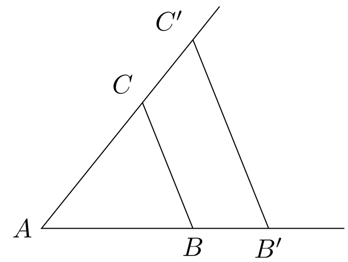
1. Similar triangles

We say that triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar with coefficient k if $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $\angle C = \angle C'$ and

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = k.$$

We will use notation $\triangle ABC \sim \triangle AB'C'$.

Theorem 21. *Consider a triangle $\triangle ABC$ and let $B' \in \overrightarrow{AB}$, $C' \in \overrightarrow{AC}$ be such that lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel. Then $\triangle ABC \sim \triangle A'B'C'$.*



The proof of this theorem is actually quite hard. For this reason, we will not give it here; in most high school geometry courses, it is taken as an axiom.

Theorem 22. *For any triangle $\triangle ABC$ and a real number $k > 0$, there exists a triangle $\triangle A'B'C'$ similar to $\triangle ABC$ with coefficient k.*

Theorem 23 (Similarity via AA)**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that $m\angle A = m\angle A'$, $m\angle B = m\angle B'$. Then these triangles are similar.*

Proof. Let $k = \frac{A'B'}{AB}$. Construct a triangle $\triangle A''B''C''$ which is similar to $\triangle ABC$ with coefficient k. Then $A'B' = A''B''$, and $m\angle A = m\angle A' = m\angle A''$, $m\angle B = m\angle B' = m\angle B''$. Thus, by ASA, $\triangle A'B'C' \cong \triangle A''B''C''$. □

Theorem 24 (Similarity via SAS)**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that $\angle A = \angle A'$, $\frac{A'B'}{AB} = \frac{A'C'}{AC}$. Then these triangles are similar.*

Theorem 25 (Similarity via SSS)**.** *Let $\triangle ABC$, $\triangle A'B'C'$ be such that*

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$$

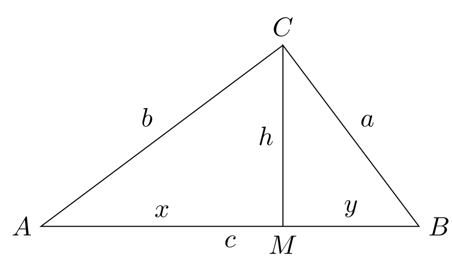
Then these triangles are similar.

One of the most important applications of the theory of similar triangles is to the study of right triangles and Pythagorean theorem (which we will discuss next time)

A right triangle is a triangle in which one of the angles is a right angle. A hypotenuse is the side opposing the right angle; two other sides are called legs.

Theorem 26. *Let $\triangle ABC$ be a right triangle, with $\angle C$ being the right angle. Let CM be the altitude of angle C. Then triangles $\triangle ABC$, $\triangle ACM$, $\triangle CBM$ are all similar.*

Proof. It immediately follows from AA similarity rule. □



This theorem immediately implies a number of important relations between various lengths in these triangles. We will give one of them. Denote for brevity $a = BC$, $b = AC$, $c = AB$, $x = AM$, $y = MB$, $h = CM$. Then we have $x : h = b : a$, $y : h = a : b$, so

$$\frac{x}{h} \times \frac{y}{h} = 1$$

or $xy = h^2$.

Homework

1. Problems 25, 26, 32 on pages pages 230-231 in the textbook.

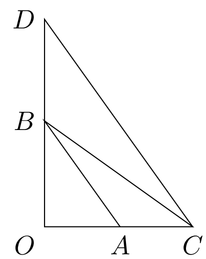
2. Show that two equilateral triangles are similar.

3. Show that two right isosceles triangles are similar.

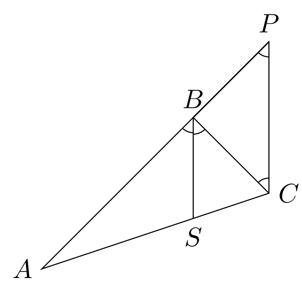
4. Triangles ABC and $A'B'C'$ are similar. The angle bisectors of angles A and A' intersect BC and $B'C'$ at D and D'. Show that $\triangle ABD \sim \triangle A'B'D'$.

5. Let $ABCD$ be a trapezoid with bases $AD = 9$, $BC = 6$, such that the height (distance between the bases) is equal to 5. Let O be the intersection point of lines AB, CD.
(a) Show that triangles $\triangle OBC$, $\triangle OAD$ are similar and find the coefficient.
(b) Find the distance from O to AD (i.e., length of the perpendicular).

6. In a triangle $\triangle ABC$, let D be midpoint of side BC, E – midpoint of side AC, F – midpoint of side AB. Prove that $\triangle DEF$ is similar to triangle $\triangle ABC$ with coefficient 1/2.



7. Consider the given diagram, where $\triangle OAB \sim \triangle OBC \sim \triangle OCD$. If $\frac{OB}{OA} = \frac{7}{5}$, is this enough information to determine $\frac{OD}{OA}$? Explain your answer.



8. Use the folowing figure to prove that an angle bisector in a triangle $\triangle ABC$ divides the opposite side in the same proportion as the two adjoining sides: $\frac{x}{y} = \frac{BA}{BC}$.
In the figure, S is the intersection of the angle bisector of $\angle B$ with the side AC, and P is a point on the line $\overleftrightarrow{AB}$ such that $BP = BC$ and P is outside $\triangle ABC$, as shown. [Hint: can you prove that all four marked angles are congruent?]