MATH 6: EUCLIDEAN GEOMETRY 2

FEB 2, 2020

1. Parallel and perpendicular lines

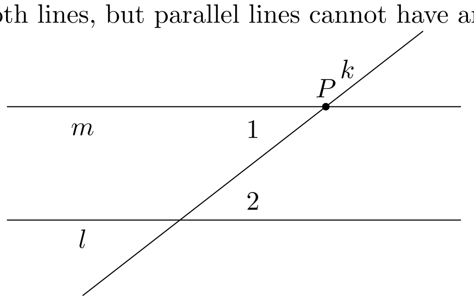
Theorem 6. *Given a line l and point P not on l, there exists exactly one line m through P which is parallel to l.*

Proof. **Existence.**

Let us draw a line k through P which inersects l. Now draw a line m through P such that alternate interior angles are equal: $m\angle 1 = m\angle 2$. Then, by Axiom 4 (alternate interior angles), we have $m \parallel l$.

Uniqueness.

To show that such a line is unique, let us assume that there are two different lines, m_1, m_2 through P both parallel to l. By Theorem 2, this would imply $m_1 \parallel m_2$. This gives a contradiction, because P is on both lines, but parallel lines cannot have any points in common, by definition!



□

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

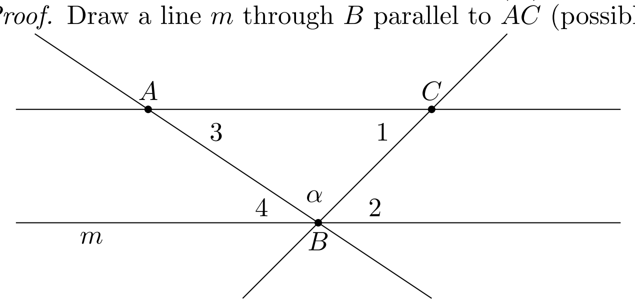
2. Sum of angles of a triangle

Definition 1. A triangle is a figure consisting of three distinct points A, B, C (called vertices) and line segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$. We denote such a triangle by $\triangle ABC$.

Similarly, a quadrilateral is a figure consisting of 4 distinct points A, B, C, D and line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ such that these segments do not intersect except at A, B, C, D.

Theorem 8. *The sum of measures of angles of a triangle is* $180°$.

Proof. Draw a line m through B parallel to $\overleftrightarrow{AC}$ (possible by Theorem 6).



[By the way: α is a Greek letter, pronounced "alpha"; mathematicians commonly use Greek letters to denote angles]

Then $m\angle 2 = m\angle 1$ as alternate interior angles, and $m\angle 4 = m\angle 3$, also alternate interior angles. On the other hand, by Axiom 3 (angles add up), we have

$$m\angle 4 + m\angle\alpha + m\angle 2 = 180°$$

Thus, $m\angle A + m\angle B + m\angle C = 180°$. □

Theorem 9. *For a triangle $\triangle ABC$, let D be a point on continuation of side AC, so that C is between A and D. Then $m\angle CBD = m\angle A + m\angle B$. (Such an angle is called the exterior angle of triangle ABC.)*

Theorem 10. *Sum of angles of a quadrilateral is equal to* $360°$.

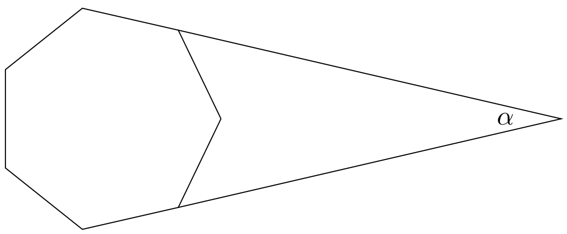
Homework

1. Prove Theorem 7.

2. Prove Theorem 9.

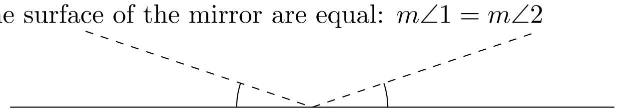
3. Deduce a formula for the sum of angles in a polygon with n vertices.

4. In the figure below, all angles of the 7-gon are equal. What is angle α?

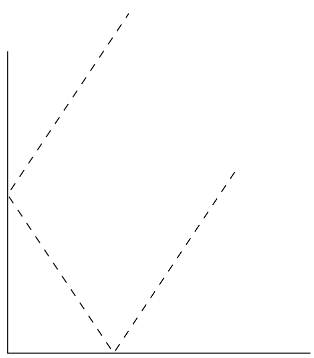


5. Show that if, in a quadrilateral $ABCD$, diagonally opposite angles are equal ($m\angle A = m\angle C$, $m\angle B = m\angle D$), then opposite sides are parallel. [Hint: show first that $m\angle A + m\angle B = 180°$.]

6. The reflection law states that the angles formed by the incoming light ray and the reflected one with the surface of the mirror are equal: $m\angle 1 = m\angle 2$



Using this law, show that a corner made of two perpendicular mirrors will reflect any light ray exactly back: the reflected ray is parallel to the incoming one:



This property – or rather, similar property of corners in 3-D – is widely used: reflecting road signs (including stop signs), tail lights of a car, reflecting strips on clothing are all contrsructed out of many small reflecting corners so that they reflect the light of a car headlamp exactly back to the car.

***7.** (Stop!)

(a) Suppose A, B, C, D are points such that $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$ and $\angle DAB \cong \angle ABC \cong \angle BCD \cong \angle CDA$. Prove that each of the four angles is a right angle. What is this figure called?

(b) Suppose X, Y are on $\overline{AB}$ (with X closer to A and Y closer to B, as shown in the figure below), and E is on $\overline{AD}$ and F on $\overline{BC}$. Prove that $m\angle EXY > 90°$.

(c) Suppose $m\angle EXY = 135°$. Prove that $\triangle EAX$ is isosceles.

(d) Suppose now that $m\angle EXY = 135°$, $\triangle EAX \cong \triangle FBY$, and $\overline{EX} \cong \overline{XY}$. Suppose also that two triangles are constructed around vertices C and D as well, essentially forming two more corner triangles, each congruent to $\triangle EAX$. Now cut all four of these triangles out from $ABCD$. What are you left with?