MATH 6: ASSIGNMENT 3: INVARIANTS

An invariant is something that does not change...

In this class (category) of problems you are given a set of objects (like some numbers) and then some operations that can be performed on these objects. The question is if a new object can be obtained from the given set of objects. For each problem, think about an invariant (a rule, an expression) that doesn't change when the operations are performed. Does the new object satisfy the invariant rule? If no, then you proved that that the new object can not be obtained from the given set. Problems 1 to 3 are also invariant problems but you are asked to predict the final result.

Here is one basic example: Suppose two parallel lines drawn on a plane. A small mouse is on one of the lines, and is allowed to walk around, as long as it stays on the lines. Is it possible for the mouse to start on one line and end up on the other line? In this case, although the mouse is allowed to walk around rather freely, it is still confined to the line it starts on - the line the mouse is on is an invariant, and ending up on the other line would change this invariant, which is impossible.

Another example: The numbers 1 through 6 are written on the board, and I am allowed to add 1 to two of the numbers at a time. Is it possible for me to repeat this action in such a way that all the numbers will end up the same? To illustrate briefly, let's say I focus on the numbers 1 through 3, and I decide to add 1 to 1 and 2, then add 1 to 1 and 2 again, and then add 1 to 1 and 3 - this will end up with the numbers 4, 4, 4, which are all the same. Can I continue this to make all the numbers the same? The answer is no, because whenever I perform this operation, the sum of the numbers increases by 2, so if the sum of the original numbers is odd, then the sum of the numbers after any step is also odd. The sum of 1 through 6 is 21, which is odd, but the sum of six identical numbers is even, therefore I can't make all the numbers the same.

1. Numbers 1 through 20 are written on the blackboard. Every minute two of the numbers are erased and replaced by their sum. Can you predict which number will be written on the board at the end?

2. Students have written on the blackboard 2011 "+" signs and 2011 "-" signs. Every minute a pair of signs is erased and replaced by a single "+" if they were equal or a single "-" if they were different. Can you predict which sign will be written on the board at the end?

3. Numbers 1 through 20 are written on the blackboard. Every minute a pair of numbers a, b are erased and replaced by $a+b-1$. Can you predict which number will be written on the board at the end?

4. There are 16 glasses on the table, one of them upside down. You are allowed to turn over any 4 glasses at a time. Can you get all glasses standing correctly by repeating this operation?

5. There are 16 glasses on a table, arranged in a 4x4 grid, the glass in the bottom-left corner upside down. You are allowed to turn over any 2x2 square of glasses at a time. Can you get all the glasses standing correctly except the one in the top-right corner?

6. Is it possible for 17 people to be Snapchat friends with each other in such a way that each person is friends with exactly three other people in the group? [Hint: how many friendships would there be?]

***7.** In the country of RGB, there are 13 red, 15 green and 17 blue chameleons. Whenever two chameleons of different colors meet, both of them change their color to the 3rd one (e.g., if red and green meet, they both turn blue). Do you think it can happen that after some time, all chameleons become the same color?[Hint: give each color a numeric value, say 0, 1, 2]

***8.** A band of four thieves are wandering a city that has a perfect square grid road plan. The thieves always face along the direction of one of the grid lines; at any time, they may walk forward to the next intersection in the direction they are facing, or they may turn 90 degrees to face a different direction. Whenever they perform one of these actions, they plunder a silver coin.
The thieves are initially positioned at the corners of downtown, which is itself the shape of a square, and begin with no silver coins. Is it possible for them to meet up at the same intersection facing the same direction in such a way that they plunder a total of an odd number of silver coins?

***9.** I decide to play a game with a pile of 20 stones, just for fun. In this game, I can split the pile into two smaller piles, and record the result on my blackboard as follows: I begin by putting the number '0' on my blackboard, and when I split the stone pile, I multiply the sizes of the resulting piles and add that number to my blackboard's number. After doing the first split, I then choose one of the remaining piles, and perform the same operation - split it into smaller piles and multiply the product of the sizes of the resulting piles and add that into the number on my blackboard. Here's an example: I can start by splitting (20) into $(5, 15)$, giving me 75 on the blackboard; then I split the 15 to get $(5, 3, 12)$, giving me $75 + 36 = 111$ on my blackboard; then split the 5 to get $(1, 4, 3, 12)$, giving me $111 + 4 = 115$ on my blackboard; etc.
I continue this game until I'm left with 20 piles of size 1. Is there a strategy I can adopt to maximize the final number that I get on my blackboard? [Hint: try playing with smaller starting pile sizes and see if you notice a pattern.]

***10.** There are 16 glasses on a table, arranged in a 4x4 grid, the glass in the bottom-left corner upside down. You are allowed to turn over any 1x4 row or column of glasses at a time. Can you get all the glasses standing correctly except the one in the top-right corner?