MATH 6
WINTER PROBLEMS

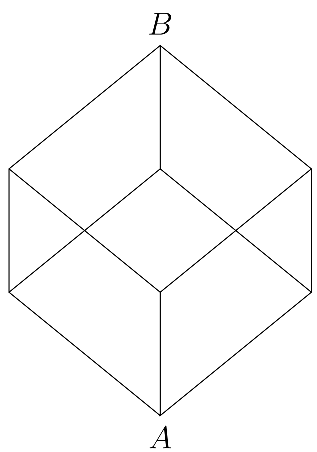
Please do one of the following problems over the winter break. See you in January!

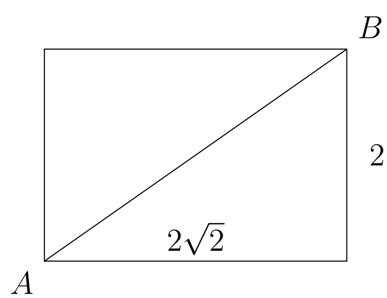
1. (Cube Problem) Given a cube of side length 2, and two opposite corners A and B of this cube,

(a) Determine the distance from A to B *(hint: you'll need the Pythagoras Theorem)*

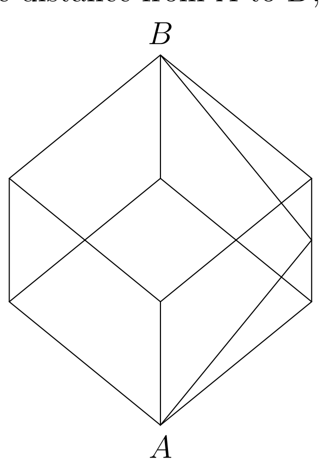
(b) Determine the length of the shortest path from A to B that does not go through the interior of the cube (such a path will travel along the faces of the cube)

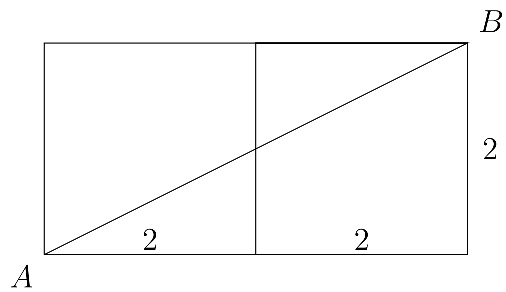
(c) Given a shortest path from A to B as in part (b), let M be intersection of this path with an edge of the cube (such a path will only intersect one edge of the cube, not including at points A and B). Determine the area of triangle $\triangle AMB$.





For (a) we use Pythagoras Theorem, noting that the diagonal of a face of the cube is $\sqrt{2}$ (also deducible using Pythagoras Theorem), and the side length is 2; letting d be the distance from A to B, we get $d^2 = (2\sqrt{2})^2 + 2^2 = 8 + 4 = 12$, so $d = \sqrt{12} = 2\sqrt{3}$.





For (b) we can consider just two faces of the cube, and unfold them onto 2 dimensions so that they form a rectangle, as shown. In this case, we see that by Pythagoras theorem, the length of the path, call it l, should satisfy $l^2 = 4^2 + 2^2 = 16 + 4 = 20$, so $l = \sqrt{20} = 2\sqrt{5}$.

For (c) we notice that the distance A to M is half the length of the path from part (b), so it should be $\sqrt{5}$. The distance M to B is the same by symmetry, thus $\triangle AMB$ is an isosceles triangle with side lengths $\sqrt{5}$, $\sqrt{5}$, $2\sqrt{3}$. One notices that the line segment from M to the midpoint of AB is the height of the isosceles triangle (this is a property of isosceles triangles), and the length of this segment, call it h, should thus satisfy $(\sqrt{5})^2 = h^2 + (\sqrt{3})^2$, i.e. $h = \sqrt{5-3} = \sqrt{2}$. The area of the triangle is then $\frac{1}{2}bh$, where b was taken to be side AB, thus we get $\frac{1}{2}(\sqrt{2})(2\sqrt{3}) = \sqrt{6}$.

2. (Arithmetic Sequence Problem) Let a_n be an arithmetic sequence with positive common difference $d_a > 0$. Let b_n be an arithmetic sequence such that $b_n^2 < a_n$ for all positive integers n. Prove that $d_b = 0$.

Firstly, the intuition for this problem is that a sequence of squares must eventually be larger than an arithmetic sequence, no matter how big the common difference is. Now, to solve the problem, first we write $a_n = a_0 + nd_a$ (where $a_0 = a_1 - d_a$) and $b_n = b_0 + nd_b$. Now, we square b_n to get

$$(b_n)^2 = (b_0 + nd_b)^2 = b_0^2 + 2nb_0d_b + n^2d_b^2$$

(Keep track of the n's, they are the most important part here.) We prove by contradiction that $d_b = 0$. Suppose $d_b \neq 0$, then we will prove that there is some n such that $(b_n)^2 > a_n$. To do this, we need to find n that satisfies

$$b_0^2 + 2nb_0d_b + n^2d_b^2 > a_0 + nd_a$$

$$b_0^2 + 2nb_0d_b + n^2d_b^2 - a_0 - nd_a > 0$$

We simplify the left hand side as follows:

$$\begin{aligned} & b_0^2 + 2nb_0d_b + n^2d_b^2 - a_0 - nd_a \\ &= n^2d_b^2 + 2nb_0d_b - nd_a - a_0 + b_0^2 \\ &= n(nd_b^2 + 2b_0d_b - d_a) - a_0 + b_0^2 \end{aligned}$$

So we need to prove that $n(nd_b^2 + 2b_0d_b - d_a) - a_0 + b_0^2 > 0$ for some n. We do this in two steps.

First we find m such that $(md_b^2 + 2b_0d_b - d_a) > 0$. Notice that since this quantity only increases with m, as d_b^2 must be positive if d_b is nonzero, and the other numbers are constants, eventually it must be positive as m increases; in particular we can solve $md_b^2 > d_a - 2b_0d_b$, to get $m > (d_a - 2d_0d_b)/(d_b)^2$, then selecting any integer that satisfies this inequality will be good enough for m, the exact number doesn't matter.

We can now proceed to step 2. Let $x = (nd_b^2 + 2b_0d_b - d_a)$, then we need to prove that $nx - a_0 + b_0^2 > 0$ for some n, noticing that for $n > m$, we have $x > 0$. Also we know that x only increases as n increases, so we can replace x by $y = (md_b^2 + 2b_0d_b - d_a)$, noting that for $n > m$, we have $x > y$. Thus it will suffice to prove that $ny - a_0 + b_0^2 > 0$ for some n, where y, a_0, and b_0^2 are all constants, and $y > 0$. Again, we have n multiplied by a positive constant and added to some integer, eventually for large enough n this must be positive; in particular we can solve $ny > a_0 - (b_0)^2$ to get $n > (a_0 - (b_0)^2)/y$.

Recall that we needed $n > m$, so to select a suitable n, we pick some n that is larger than both $(a_0 - (b_0)^2)/y$ and m. Doing this, we have found a value of n such that $b_n^2 > a_n$, which is a contradiction to the requirement that $b_n^2 < a_n$ for all n. We deduce that $d_b \neq 0$ must be false, i.e. $d_b = 0$ must be true.