MATH 6: ASSIGNMENT 1

JANUARY 5, 2020

1. Geometric Sequences

A sequence of numbers is a geometric sequence if the next number in the sequence is the current number times a constant, called the **common ratio**. let's call it q. For example, let's consider the sequence: 6, 12, 24, 48, ... Notice that the common ratio is 2, q =2. The flrst term in the sequence is b_1 =6, the sccond is $b_2 = 6 \cdot 2$ =12, $b_3 = 12 \cdot 2$. The common ratio is $q = 2$. What is the n^{th} term? For example what is b_{10}?

$b_1 = 6$
$b_2 = 6 \cdot 2 = 12$
$b_3 = (6 \cdot 2) \cdot 2 = 6 \cdot 2^2 = 24$
$b_4 = (6 \cdot 2^2) \cdot 2 = 6 \cdot 2^3 = 48$
...
In general, the n^{th} term b_n is equal to $b_1 \cdot q^{n-1}$

2. A Property of Geometric Sequences

A property of a geometric sequence is that any term is the geometric mean of its neighbors. For example, $b_2 = \sqrt{b_1 \cdot b_3} = \sqrt{6 \cdot 24} = 12$. In general,

$$b_n = \sqrt{b_{n-1} \cdot b_{n+1}}$$

3. Sum of a Geometric Sequence

Let's try to sum $1+2+4+...+64$. For purposes of working with this sum, let it be called S, i.e. $S = 1+2+4+...+64$. Then I can notice that $2S = 2+4+8+...+128$; subtract the original sum to get $2S - S = 128 - 1$ (everything else cancels out). Thus $S = 127$. What did we do here? We multiplied by 2, which lined up the terms of the sequence to the next term over. In the geometric sequence 1, 2, ..., 64, the common ratio is $q = 2$.
Let's do this in general. Let b_1, ... b_n be a geometric sequence with common ratio q. Then,

$$S = b_1 + b_2 + ... + b_n$$

$$q \cdot S = q \cdot b_1 + q \cdot b_2 + ... + q \cdot b_n = b_2 + b_3 + ... + b_{n+1}$$
$$q \cdot S - S = b_{n+1} - b_1 = b_1 \cdot q^n - b_1$$
$$S \cdot (q-1) = b_1 \cdot (q^n - 1)$$
$$S = b_1 \cdot \frac{q^n - 1}{q - 1}$$

4. Homework

1. Write out the first four terms of each of the following geometric sequences, given the first term b_1 and common ratio q.

(a) $b_1 = 1$ and $q = 3$
(b) $b_1 = 1$ and $q = \frac{1}{2}$
(c) $b_1 = -10$ and $q = \frac{1}{2}$
(d) $b_1 = 27$ and $q = -\frac{1}{3}$

2. Calculate $S = 1 + 3 + 9 + 27 + 81 + 243$, first via the method of multiplying by the common ratio, then by plugging into the formula directly. Which method do you like better?

3. Calculate $S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$, using your preferred method.

4. What are the first two terms of the geometric sequence b_1, b_2, 24, 36, 54 ... ? Remember that you can find the common ratio by dividing a term by the previous term.

5. What is the common ration of the geometric sequence $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots$? What is b_{10}? b_{99}? b_{100}?

6. Calculate the sum $1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + \ldots - 2^{15}$

***7.** Let $b_1, b_2, \ldots, b_5$ be a geometric sequence. Is it possible for these numbers to also form an arithmetic sequence? Are there any values of common ratio q that would be impossible (i.e., such numbers with common ratio q could never be an arithmetic sequence)?

***8.** Let $b_1 \ldots b_n$ be a geometric sequence with $q > 1$. Let $c_1 = b_2 - b_1$, $c_2 = b_3 - b_2$, ... $c_{n-1} = b_n - b_{n-1}$. What kind of sequence is $c_1 \ldots c_n$?

***9.** Calculate the sum $1 - 2 + 2^2 + 2^3 - 2^4 + 2^5 + 2^6 - 2^7 + 2^8 + \ldots + 2^{14}$