MATH 6: HOMEWORK 9

Permutations with Repetitions

If there are identical objects among the selections, then there will be some overcounting that we need to correct. We do that by dividing to the number of arrangements of the repeated objects. Let's take a look at the number of ways to arrange the letters in the word WALL. If the letters would be distinct, then the number of arrangements (permutations) would be 4!. But because the letter L is repeated, we need to divide by 2!, the number of ways to arrange 2 objects (both L letters). The answer is $\frac{4!}{2!} = 12$. Another examplehow many different arrangements can be formed with the letters from the word ALLELE? The answer is: $\frac{6!}{3!2!} = 20$. We divide by 3! because the three letters L are indistinguishable, and 2! for the Es.

Combinations

Combinations are used when order does not matter. For example when a sports team is selected, or a committee is formed, it does not matter which team member is selected first or second or third. What matters is only the fact that they are selected. We can think of this as arrangements that need to be corrected because of overcounting - essentially, everyone on the selected team is 'identical'. Combinations can be written as $\binom{n}{k}$ for k objects chosen out of n objects, we say "n choose k". Combinations can be also written as ${}_n\mathrm{C}_\mathrm{k}$. The two notations are equivalent.

$$ {}_n\mathrm{C}_\mathrm{k} = \binom{\mathrm{n}}{\mathrm{k}} = \frac{\mathrm{n}!}{\mathrm{k}!(\mathrm{n}-\mathrm{k})!} = \frac{{}_\mathrm{n}\mathrm{P}_\mathrm{k}}{\mathrm{k}!} $$

Lastly, remember that $n \cdot (n-1)! = n!$, and in general $n \cdot (n-1) \cdot ... \cdot (k+1) \cdot k! = n!$

Homework

1. (a) List all arrangements of letters in the word WALK.
(b) Copy the same list again, except replace all the letters K with the letter L.
(c) Cross out any repeated arrangements in your second list (that is, if you find two identical arrangements of letters, you may cross one out).
(d) Explain why the number of ways to arrange the letters of WALK is double the number of ways to arrange the letters of WALL.

2. Compute $\binom{6}{3}$, $\binom{8}{5}$, $\binom{10}{0}$.

3. In how many ways can you arrange the letters from the word TICKTOCK?

4. In how many ways can you select a team of 3 from 8 people?

5. Show that $\binom{n}{k} = \binom{n}{n-k}$

6. How many triangles can be formed by connecting the vertices of a hexagon?

7. How many paths can you form on a grid with 3 rows and 4 columns if you start at the lower left corner and finish at the upper right corner? You can go only on the lines of the grid and you can only go to the right or up.

8. In December of 2019, The Great Amphibian Parliament must send a delegation to the International Freshwater Convention. The IFC requires that 6 delegates be sent, and the GAP wishes to send 4 frogs and 2 salamanders. If there are 28 frog MPs and 12 salamander MPs, how many ways can the GAP choose a delegation to send to the IFC?

9. In December of 2020, the Great Amphibian Parliament must send a delegation to the International Freshwater Convention, and a single delegate to the Estuary Summit. The 2020 IFC requires only three delegates, so the GAP decides they are willing to send any three of their MPs. In how many ways can the GAP choose one delegate to go to the ES and three to go to the IFC?

10. Prove the following for any natural numbers n, r, t (you may assume $r < n$):

$$n^t \binom{n-1}{r}^t = (n-r)^t \binom{n}{r}^t \tag{1}$$

$$\binom{tn}{1}\binom{n}{r}^{t-1}\binom{n-1}{r} = \binom{t(n-r)}{1}\binom{n}{r}^t \tag{2}$$

11. Prove that if $a^2 + b^2 = 1$, then $(a^2 - b^2)^2 + (2ab)^2 = 1$