MATH 10
ASSIGNMENT 21: GROUPS

APR 26, 2020

Definition. A *group* is a set G with a binary operation $*$ and a special element e such that teh following properties hold:

1. Associativity: $(a * b) * c = a * (b * c)$
2. Unit: there is an element $e \in G$ such that for any $g \in G$, we have $e * g = g * e = g$
3. Inverses: for any $g \in G$, there exists an element $h \in G$ such that $g * h = h * g = e$

The operation in groups is also commonly written as a dot (e.g. $g \cdot h$) or without any sign at all (e.g. gh). The unit element is sometimes denoted just 1, and the inverse of g by g^{-1} (see problem 2 below)

A typical example of a group is the group of all permutations of the set $\{1, \ldots, n\}$. It is commonly denoted S_n and called the *symmetric group*. More examples are given in problem 1 below.

Homework

1. Show that the following are groups:
   (a) Set $\mathbb{Z}$ with the operation of addition
   (b) Set $\mathbb{R}$ with the operation of addition
   (c) Set $\mathbb{R}^\times = \mathbb{R} - \{0\}$ with the operation of multiplication
   (d) Set of all vectors in 3 dimensional space, with the operation of addition.
   (e) Set $\mathbb{Z}_n$ of all integers modulo n with the operation of addition modulo n.
   (f) Matrices of the form
   $$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix},$$
   with the operation of matrix product
   (g) Set O_3 of all rigid motions (i.e., transformations preserving distances) of the 3-dimensional space, with the operation of composition.

2. Prove that in a group, each element g has a *unique* inverse: there is exactly one h such that $gh = hg = e$. (Note that the definition of the group only requires that such an h exists and says nothing about uniqueness). Hint: if h_1, h_2 are different inverses, what is $h_1 g h_2$?

3. Prove that in any group, $(xy)^{-1} = y^{-1}x^{-1}$

4. Consider the set D_n of all symmetries of a regular n-gon (a symmetry is a transformation of the plane that preserves distances and which sends the regular n-gon into itself). Prove that D_n is a group with respect to composition. How many elements are there in D_n? How many of them are rotations?

5. Consider the set R of all rotations of 3-dimensional space which preserve a regular tetrahedron.
   (a) How many elements are there in R?
   (b) Prove that R is a group.
   *(c) Every element of R permutes vertices of the tetrahedron and thus determines an element of S_4. Show that this allows one to identify R with the group A_4 of even permutations of 4 elements.