MATH 10
ASSIGNMENT 20: INTERMEDIATE VALUE THEOREM
APR 5, 2020

Definition. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called continuous if, for every sequence $a_n \in \mathbb{R}$ which has a limit: $\lim a_n = A \in \mathbb{R}$, the sequence $f(a_n)$ also has a limit and $\lim f(a_n) = f(A)$.

It was proved last time that the sum and product of continuous functions is continuous; the same is true for f/g as long as $g \neq 0$. In particular, all polynomials and rational fucntions are continuous everywhere they are defined.

Theorem (Intermediate Value Theorem)**.** *Let $f(x)$ be a continuous fucntion on the interval $[a, b]$ such that $f(a) < 0$ and $f(b) > 0$. Then there exists a point $c \in (a, b)$ such that $f(c) = 0$.*

A proof was discussed in class.

Homework

1. Prove that polynomial $x^3 + 3x - 2$ has a root between 0 and 5.

2. Prove that there exists a poistive number x such that $\sin(x) = 0.5x$. (You can use without proof the fact that $\sin(x)$ is continuous).

3. Let $f(x) = x^{2n+1} + \dots$ be a polynomial of odd degree, with leading coefficient 1.
  (a) Prove that for large enough x, $f(x) > 0$. (I.e., there exists a real number M such that for all $x \geq M$, $f(x) > 0$.)
  (b) Prove that for large enough x, $f(-x) < 0$.
  (c) Porve that $f(x)$ has at least one real root.

4. A traveler leaves town A at 9 am on Monday and arrives at town B at 4 pm the same day. He spends the night at town B, leaves it at 9 am on Tuesday, and returns to town A by 4 pm on Tuesday, following the same road.

Prove that there is a point on the road which he passed at exact same time on Monday and Tuesday.

Note that we are not assuming that the traveler goes at constant speed.

5. Given a convex polygon S and a point A inside it, prove that there exists a chord of S which has A as the midpoint. [Hint: consider difference of lengths of the two pieces of a chord through A as a fucntion of the angle.]

***6.** We are given 10 red and 10 blue points in the plane, such that no three of them are on the same line. Prove that there is a line such that on each side of it there are 5 red and 5 blue points.